MIDTERM 2010

Problem 1 [25]

(a) [10]
$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = \dfrac{|x|}{x} \end{cases}$$

$$\Gamma: \begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = \dfrac{|s|}{s} \end{cases}$$

① $\dfrac{dt}{d\tau} = 1 \Rightarrow t = \tau$

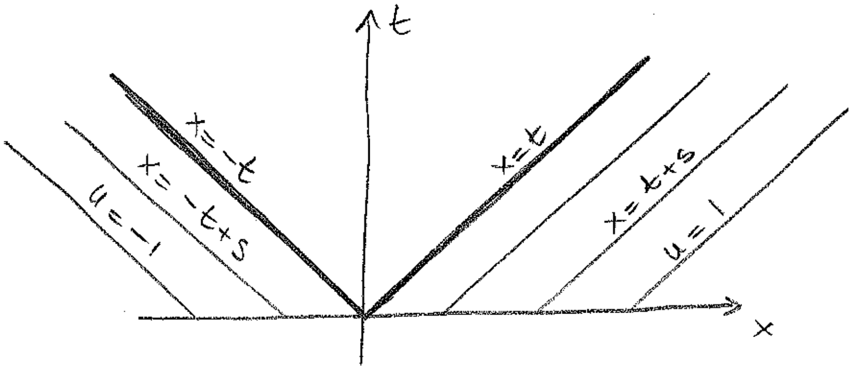
③ $\dfrac{dx}{d\tau} = u \Rightarrow x(\tau) = u_0(s)\tau + s$

② $\dfrac{du}{d\tau} = 0 \Rightarrow u = u_0(s)$

$\Rightarrow x = \dfrac{|s|}{s} t + s$. are the characteristics.

if $s > 0 \Rightarrow x = t + s$

if $s < 0 \Rightarrow x = -t + s$



$\Rightarrow$ there is an expansion shock starting at $s = 0$. in the region delimited by the $x = -t$ and $x = t$ characteristics.

In the expansion shock, using the formula from lectures

$u(x,t) = G\left(\dfrac{x-0}{t}\right)$ where G is the inverse of $F'(u) = u \Rightarrow$ Here $G(u) = u$

$$\Rightarrow \boxed{u(x,t) = \frac{x}{t}}$$

Check: on the $x = t$ characteristic $u = 1$

on the $x = -t$ characteristic $u = -1$

$\Rightarrow$ Complete solution is

$$u(x,t) = \begin{cases} -1 & \text{if} \quad x \leq -t \\ \frac{x}{t} & \text{if} \quad -t \leq x \leq t \\ 1 & \text{if} \quad x \geq t. \end{cases}$$

(b) [5] Characterising the $x = t$ and $x = -t$ line as a new initial condition curve Γ:

$$\Gamma = \begin{cases} x_0(s) = s \\ t_0(s) = |s| \end{cases}$$

<u>On this new Γ curve</u> $\quad u_0(s) = \frac{|s|}{s}$ or $u_0(s) = \frac{s}{|s|}$
(it's the same thing)

(c) [10]. $\rightarrow$ we now solve the problem again

$$\frac{dt}{d\tau} = 1 \Rightarrow t = \tau + |s|$$

$$\frac{du}{d\tau} = 0 \Rightarrow u = u_0(s) = \frac{|s|}{s}$$

$$\frac{dx}{d\tau} = u \Rightarrow x = \frac{|s|}{s}\tau + s$$

<u>note</u> $\frac{|s|}{s}|s| = \frac{s}{|s|}|s| = s$

$$\Rightarrow \quad x = \frac{|s|}{s}\left(t - |s|\right) + s$$

$$= ut - \frac{|s|^2}{s} + s = ut - s + s = ut$$

So $\boxed{u = \frac{x}{t}}$ as before.

$\rightarrow$ this shows that using the formula from lectures does recover the same result as reconstructing the solution in the shock region from the solution on the last characteristics

PROBLEM 2

(a) [4]

$$P_{n_0}(t+\Delta t) = P_{n_0}(t)\,(\text{probability that none of the } n_0 \text{ bacteria reproduce})$$

$$= P_{n_0}(t)\,(1 - n_0 \lambda \Delta t)$$

$$\Rightarrow \frac{dP_{n_0}}{dt} = \frac{P_{n_0}(t+\Delta t) - P_{n_0}(t)}{\Delta t} = -n_0 \lambda P_{n_0}$$

$n > n_0$

$$P_n(t+\Delta t) = P_n(t)\,(\text{probability that none of the } n \text{ bacteria reproduce})$$

$$+ P_{n-1}(t)\,(\underline{\text{probability that one of}} \text{ reproduce})$$

$$= P_n(t)\,(1 - \lambda n \Delta t) + P_{n-1}(t)\,\lambda(n-1)\Delta t.$$

so

$$\frac{dP_n}{dt} = \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -\lambda n P_n + \lambda(n-1) P_{n-1}$$

[6] (b) Let $G = \sum_{n=n_0}^{\infty} P_n(t)\, s^n$

$$\frac{\partial G}{\partial t} = \sum_{n=n_0}^{\infty} \frac{dP_n}{dt} s^n = \frac{dP_{n_0}}{dt} s^{n_0} + \sum_{n=n_0+1}^{\infty} \frac{dP_n}{dt} s^n$$

$$= -n_0 \lambda P_{n_0} s^{n_0} + \sum_{n=n_0+1}^{\infty} \left(-\lambda n P_n + \lambda(n-1) P_{n-1}\right) s^n$$

$$= \sum_{n=n_0}^{\infty} -n\lambda P_n s^n + \sum_{m=n_0}^{\infty} \lambda m P_m s^{m+1}$$

$$m = n-1$$

However, $\frac{\partial G}{\partial s} = \sum_{n=n_0}^{\infty} P_n(t)\, n s^{n-1}$

So

$$\frac{\partial G}{\partial t} = \sum_{n=n_0}^{\infty} -s\lambda\left(nP_n s^{n-1} - s n P_n s^{n-1}\right) = -s\lambda(1-s)\frac{\partial G}{\partial s}.$$

So $\quad \dfrac{\partial G}{\partial t} + \lambda s(1-s)\dfrac{\partial G}{\partial s} = 0 \quad$ as required

and at $t=0$ $\quad G = s^{n_0} \quad$ so $\quad G(0,s) = s^{n_0}$

[10] (c) We solve the above using the method of characteristics

$$\frac{\partial t}{\partial \tau} = 1 \quad \to t = \tau \quad ①$$

(let's use τ, x as variables on the characteristics and ICC).

$$\frac{\partial s}{\partial \tau} = \lambda s(1-s)$$

$$\frac{\partial G}{\partial \tau} = 0$$

$$\int \frac{ds}{s(1-s)} = \int \lambda d\tau$$

$$\frac{1}{s(1-s)} = \frac{A}{s} + \frac{B}{1-s}$$

with $A(1-s) + Bs = 1$

$\Leftrightarrow$ $A = 1$
$B - A = 0 \quad B = 1$

$$\text{so} \int \frac{ds}{s(1-s)} = \int \frac{ds}{s} + \frac{ds}{1-s}$$
$$= \ln|s| - \ln|1-s|$$

So $\quad \ln|s| - \ln|1-s| = \lambda\tau + \text{constant}$

② at $\tau = 0$ $\quad s = x \quad \Rightarrow \quad \ln|s| - \ln|1-s| = \lambda\tau + \ln|x| - \ln|1-x|$

Finally, $\quad G = G(0,x) = x^{n_0} \quad$ ③

Combining ①, ② and ③ we get

$$\left| \frac{s(1-x)}{x(1-s)} \right| = e^{\lambda\tau} = e^{\lambda t}$$

$$\text{So} \quad \left| \frac{(1-x)}{x} \right| = \left| \frac{1-s}{s} \right| e^{\lambda t}$$

$$\Rightarrow \quad \left| \frac{1}{x} - 1 \right| = \left| \frac{1-s}{s} \right| e^{\lambda t}$$

now by construction $0 \leq s \leq 1$ and so $0 \leq x \leq 1$

So we get $\frac{1}{x} = 1 + \left(\frac{1}{s} - 1\right)e^{\lambda t}$

$$\Rightarrow \quad x = \frac{1}{1 + \left(\frac{1}{s} - 1\right)e^{\lambda t}}$$

so $G = x^{n_0} = \left[\frac{s}{s + (1-s)e^{\lambda t}}\right]^{n_0}$

$$= s^{n_0}\left[\frac{e^{-\lambda t}}{se^{-\lambda t} + 1 - s}\right]^{n_0}$$

$$= s^{n_0}\left[\frac{e^{-\lambda t}}{1 - s(1-e^{-\lambda t})}\right]^{n_0} \quad \text{as required.}$$

[5] (d) The expectation of $X(t)$ is $E(X(t)) = \left.\frac{\partial G}{\partial s}\right|_{s=1}$

$$\frac{\partial G}{\partial s} = n_0 s^{n_0 - 1}\left[\frac{e^{-\lambda t}}{1 - s(1-e^{-\lambda t})}\right]^{n_0}$$

$$+ s^{n_0} e^{-\lambda t n_0} \frac{-(1-e^{-\lambda t})}{(1 - s(1-e^{-\lambda t}))^{n_0+1}}(-n_0)$$

$$\left.\frac{\partial G}{\partial s}\right|_{s=1} = n_0 + \frac{n_0(1-e^{-\lambda t})}{e^{-\lambda t}} = n_0 e^{\lambda t} \quad \text{as required.}$$

$$4y^2 u_{xx} + 2(1-y^2)u_{xy} - u_{yy} - \frac{2y}{1+y^2}(2u_x - u_y) = 0$$

$$a = 4y^2 \quad b = (1-y^2) \quad c = -1$$

(a) [15]

$$\delta = (1-y^2)^2 + 4y^2 = (1+y^2)^2$$

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \frac{(1-y^2) \pm \sqrt{(1+y^2)^2}}{4y^2} = \begin{cases} \frac{2}{4y^2} = \frac{1}{2y^2} \\ \frac{-2y^2}{4y^2} = -\frac{1}{2} \end{cases}$$

$$\frac{dy}{dx} = \frac{1}{2y^2} \Rightarrow 2y^2\,dy = dx \Rightarrow \frac{2}{3}y^3 = x + \xi$$

$$\boxed{\xi = \frac{2}{3}y^3 - x}$$

$$\frac{dy}{dx} = -\frac{1}{2} \Rightarrow y = -\frac{1}{2}x + \eta \Rightarrow \boxed{\eta = y + \frac{1}{2}x}$$

$$u_x = -u_\xi + \frac{1}{2}u_\eta$$

$$u_{xx} = \left(-\frac{\partial}{\partial \xi} + \frac{1}{2}\frac{\partial}{\partial \eta}\right)^2 u = u_{\xi\xi} - u_{\xi\eta} + \frac{1}{4}u_{\eta\eta}$$

$$u_y = 2y^2 u_\xi + u_\eta$$

$$u_{xy} = \frac{\partial}{\partial x}\left(2y^2 u_\xi + u_\eta\right) = 2y^2 \frac{\partial}{\partial x}(u_\xi) + \frac{\partial}{\partial x}(u_\eta)$$

$$= 2y^2\left[-u_{\xi\xi} + \frac{1}{2}u_{\xi\eta}\right] + \left[-u_{\xi\eta} + \frac{1}{2}u_{\eta\eta}\right]$$

$$= -2y^2 u_{\xi\xi} + (y^2-1)u_{\xi\eta} + \frac{1}{2}u_{\eta\eta}$$

$$u_{yy} = \frac{\partial}{\partial y}\left[2y^2 u_\xi + u_\eta\right] = 4y\,u_\xi + 2y^2\frac{\partial}{\partial y}(u_\xi) + \frac{\partial}{\partial y}(u_\eta)$$

$$= 4y\,u_\xi + 2y^2\left(2y^2 u_{\xi\xi} + u_{\xi\eta}\right) + 2y^2 u_{\xi\eta} + u_{\eta\eta}$$

$$= 4y\,u_\xi + 4y^4 u_{\xi\xi} + 4y^2 u_{\xi\eta} + u_{\eta\eta}$$

$$4y^2 u_{xx} + 2(1-y^2)u_{xy} - u_{yy} - \frac{2y}{1+y^2}(2u_x - u_y) = 0$$

$$\Rightarrow 4y^2\left(\cancel{u_{\xi\xi}} - u_{\xi\eta} + \cancel{\tfrac{1}{4}u_{\eta\eta}}\right) + 2(1-y^2)\left(-\cancel{2y^2 u_{\xi\xi}} + (y^2-1)u_{\xi\eta} + \cancel{\tfrac{1}{2}u_{\eta\eta}}\right)$$

$$-\left(4y\, u_\xi + \cancel{4y^4 u_{\xi\xi}} + 4y^2 u_{\eta\xi} + \cancel{u_{\eta\eta}}\right) - \frac{2y}{1+y^2}\left(-2u_\xi + u_\eta - 2y^2 u_\xi - u_\eta\right) = 0$$

$$\Rightarrow -4y^2 u_{\xi\eta} + 2(1-y^2)(y^2-1)u_{\xi\eta} - 4y^2 u_{\eta\xi}$$

$$-4y\,u_\xi - \frac{2y}{\cancel{1+y^2}}(-2)\cancel{(1+y^2)}\,u_\xi = 0$$

$$\Rightarrow \left[-8y^2 - 2(1-y^2)^2\right]u_{\xi\eta} = 0$$

$$\Rightarrow \boxed{u_{\eta\xi} = 0}$$

(b) [5] $\Rightarrow \quad u = F(\eta) + G(\xi) = F\left(y + \tfrac{1}{2}x\right) + G\left(\tfrac{2}{3}y^3 - x\right)$

(c) [5] $u(x,0) = g(x) \quad \Rightarrow \quad F\left(\tfrac{1}{2}x\right) + G(-x) = g(x)$

$u_y(x,0) = f(x) \quad \Rightarrow \quad F'\left(\tfrac{1}{2}x\right) = f(x)$

So $\quad F'(x) = f(2x)$

$$F(x) = \int_0^x f(2x')\,dx' + F(0)$$

$$\Rightarrow G(-x) = g(x) - \int_0^{1/2\,x} f(2x')\,dx' - F(0)$$

$$\Rightarrow G(x) = g(-x) - \int_0^{-x/2} f(2x')\,dx' - F(0)$$

$$\Rightarrow u(x,y) = \int_0^{y+\frac{1}{2}x} f(2x')\,dx' + g\left(-\tfrac{2}{3}y^3 + x\right) - \int_0^{-\frac{1}{2}\left(\frac{2}{3}y^3 - x\right)} f(2x')\,dx'$$

$$= g\left(-\tfrac{2}{3}y^3 + x\right) + \int_{-\frac{1}{2}\left(\frac{2}{3}y^3 - x\right)}^{y+2x} f(2x')\,dx'$$

Problem 4 [25]

(a) [15]

$$\frac{\partial^2 h}{\partial t^2} = c^2 \frac{\partial^2 h}{\partial x^2} - b \frac{\partial h}{\partial t}$$

$$h(0,t) = h(L,t) = 0$$

$$h(x,0) = f(x)$$

$$h_t(x,0) = g(x)$$

Separation of variables: let $h(x,t) = A(x)B(t)$

$$\Rightarrow \quad A\ddot{B} = c^2 A'' B - b\dot{B}A$$

$$\Rightarrow \quad \frac{\ddot{B} + b\dot{B}}{B} = \frac{c^2 A''}{A} = K$$

↑ both have to be constant

By inspection, A has to be either a sin or cos ⇒ let

$K = -k^2$ so $A'' = -\frac{k^2}{c^2} A$

$$A = \begin{cases} \cos\left(\frac{k}{c}x\right) \\ \sin\left(\frac{k}{c}x\right) \end{cases}$$

$A(0) = 0 \Rightarrow$ cannot be the cosine solution

$A(L) = 0 \Rightarrow L\frac{k}{c} = n\pi \Rightarrow k_n = \frac{n\pi c}{L}$

So $A_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

Going back to the B_n equation:

$$\ddot{B}_n + b\dot{B} = -\frac{n^2\pi^2c^2}{L^2} B$$

let $B_n \sim e^{\lambda_n t} \Rightarrow \lambda_n^2 + b\lambda_n + \frac{n^2\pi^2c^2}{L^2} = 0$

$$\Rightarrow \lambda_n^{+,-} = \frac{-b \pm \sqrt{b^2 - \frac{4n^2\pi^2c^2}{L^2}}}{2}$$

$\Rightarrow$ some λ_n may be real, some may be complex

but we can always write

$$B_n(t) = \alpha_n e^{\lambda_n^+ t} + \beta_n e^{\lambda_n^- t}$$

$\rightarrow$ General solution is

$$h(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left\{ \alpha_n e^{\lambda_n^+ t} + \beta_n e^{\lambda_n^- t} \right\}$$

At $t=0$:

$$h(x,0) = f(x) = \sum_{n=1}^{\infty} (\alpha_n + \beta_n) \sin\left(\frac{n\pi x}{L}\right)$$

$$h_t(x,0) = g(x) = \sum_{n=1}^{\infty} \left(\alpha_n \lambda_n^+ + \beta_n \lambda_n^-\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \quad \alpha_n + \beta_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\alpha_n \lambda_n^+ + \beta_n \lambda_n^- = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

So

$$\alpha_n = \frac{1}{\lambda_n^- - \lambda_n^+}\left\{ \frac{2\lambda_n^-}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx - \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \right\}$$

$$\beta_n = \frac{1}{\lambda_n^+ - \lambda_n^-}\left\{ \frac{2\lambda_n^+}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx - \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \right\}$$

Note that in some cases (when $\lambda_n \in \mathbb{C}$) then α_n & β_n are also complex, while when $\lambda_n \in \mathbb{R}$ then α_n and β_n are real.

(b) [10].

$$f(x) = 0$$

$$g(x) = \begin{cases} \dfrac{2V_0 x}{L} & \text{if } x \in [0, L/2] \\ 2V_0 - \dfrac{2V_0 x}{L} & \text{if } x \in [L/2, L] \end{cases}$$

$\Rightarrow$ we need to calculate $\displaystyle\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$

$$\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx = \int_0^{L/2} 2V_0 \frac{x}{L} \sin\left(\frac{n\pi x}{L}\right)dx + \int_{L/2}^{L}\left(2V_0 - \frac{2V_0 x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2V_0}{L}\left[-\frac{L}{n\pi} x \cos\left(\frac{n\pi x}{L}\right)\right]_0^{L/2} + \frac{2V_0}{L}\frac{L}{n\pi}\int_0^{L/2}\cos\left(\frac{n\pi x}{L}\right)dx$$

$$+ 2V_0\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_{L/2}^{L} - \frac{2V_0}{L}\left[-\frac{L}{n\pi} x \cos\left(\frac{n\pi x}{L}\right)\right]_{L/2}^{L}$$

$$- \frac{2V_0}{L}\frac{L}{n\pi}\int_{L/2}^{L}\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2V_0}{L}\left(\cancel{-\frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)}\right) + \frac{2V_0}{L}\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$- 2V_0\frac{L}{n\pi}\left(\cancel{\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)}\right) + \frac{2V_0}{L}\frac{L}{n\pi}\left(\cancel{L\cos(n\pi)} - \cancel{\frac{L}{2}\cos\left(\frac{n\pi}{2}\right)}\right)$$

$$+ \frac{2V_0}{L}\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right).$$

$$= \frac{4V_0 L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \alpha_n = \frac{2}{L}\cdot\frac{4V_0 L}{n^2\pi^2}\cdot\frac{1}{\sqrt{b^2 - \frac{4n^2\pi^2c^2}{L^2}}} = \frac{8V_0}{n^2\pi^2}\frac{1}{\sqrt{b^2 - \frac{4n^2\pi^2c^2}{L^2}}}$$

$$\beta_n = -\alpha_n = -\frac{8V_0}{n^2\pi^2}\frac{1}{\sqrt{b^2 - \frac{4n^2\pi^2c^2}{L^2}}} \qquad (*)$$

$$\Rightarrow h(x,t) = \sum_{n=1}^{\infty}{}' \sin\left(\frac{n\pi x}{L}\right)\frac{8V_0}{n^2\pi^2}\frac{e^{-b/2\, t}}{\sqrt{b^2 - 4n^2\pi^2c^2/L^2}}\cdot 2\sinh\left(\sqrt{b^2 - \frac{4n^2\pi^2c^2}{L^2}}\,\frac{t}{2}\right)$$

> Note: recall that $\sin(x) = \dfrac{\sinh(ix)}{i}$, so this expression is always real