3.3.3 Canonical form for Elliptic equations

Given a second order linear PDE which is elliptic, to reduce it to its canonical form we must find a coordinate change $(x,y) \to (\xi,\eta)$ such that

$$\begin{cases} A = C \\ B = 0 \end{cases}$$

So we need
$$\begin{cases} a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 \\ a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = 0 \end{cases}$$

let's construct the complex quantity $\phi = \xi + i\eta$ then this system is equivalent to

$$a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 = 0$$

Indeed

$$a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 = a(\xi_x + i\eta_x)^2 + 2b(\xi_x + i\eta_x)(\xi_y + i\eta_y) + c(\xi_y + i\eta_y)^2$$

$$= a\xi_x^2 - a\eta_x^2 + 2b(\xi_x\xi_y - 2b\eta_x\eta_y + c\xi_y^2 - c\eta_y^2 + i\left[2a\xi_x\eta_x + 2b(\xi_x\eta_y + \eta_x\xi_y) + 2c\,\xi_y\eta_y\right]$$

So equating real & imaginary parts to 0 recovers the required system.

$\Rightarrow$ Characteristic equations imply

$$\frac{dy}{dx} = \frac{b \pm i\sqrt{ac-b^2}}{a} \qquad \text{since } ac - b^2 < 0$$

however, this time the characteristics "live" in a "complex plane".

The characteristic equations are complex conjugates so their solutions (say ϕ and ψ) will also be C.Cs.

Once the solution is found, we recover ξ and η by taking

$$\xi = \mathrm{Re}(\phi)$$
$$\eta = \mathrm{Im}(\phi)$$

(Note: we can arbitrarily choose ϕ or ψ → the only difference is in the sign of η).

Example: the Tricomi equation $u_{xx} + x u_{yy} = 0$ for $x > 0$

then we solve

$$\frac{dy}{dx} = \pm i\sqrt{x} \Rightarrow dy = \pm i\sqrt{x}\, dx$$

so the solution is

$$\frac{3}{2} y = \pm i x^{3/2} + \text{constant}$$ → choose constant $= \phi$

so let $\phi = \frac{3}{2} y \pm i x^{3/2}$

so
$$\begin{cases} \xi = \frac{3}{2} y \\ \eta = x^{3/2} \end{cases}$$

then
$$\begin{cases} \xi_x = 0 & \xi_y = \frac{3}{2} \\ \eta_x = \frac{3}{2} x^{1/2} & \eta_y = 0 \qquad \eta_{xx} = \frac{3}{4} x^{-1/2} \end{cases}$$

so
$$u_{xx} + x u_{yy} = \frac{9}{4} x\, u_{\eta\eta} + \frac{3}{4} x^{-1/2} u_\eta + x\left(\frac{9}{4} u_{\xi\xi}\right)$$
$$= 0$$

$$x = \left(\frac{2}{3}\eta_x\right)^2$$

$$\Rightarrow u_{\eta\eta} + u_{\xi\xi} + \frac{1}{3} x^{-3/2} u_\eta = 0$$

$$\Rightarrow u_{\xi\xi} + u_{\eta\eta} + \frac{1}{3\eta} u_\eta = 0$$ → Canonical form of the equation for $x > 0$

SUMMARY

When trying to find the canonical form of

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + \mathcal{L}^{(1)}(u) = g(x,y)$$

① Construct $\dfrac{dy}{dx} = \dfrac{-b \pm \sqrt{b^2 - ac}}{a}$, and solve this ODE

② • if $b^2 - ac > 0$ then we get 2 equations, yielding two solutions ξ and η.

• if $b^2 - ac = 0$ then we get 1 equation for η. Then choose any ξ such that the mapping $(x,y) \to (\xi, \eta)$ is indeed a change of coordinates

• if $b^2 - ac < 0$ then we get two complex conjugate solutions ϕ and ϕ^*. Then

$$\xi = \mathrm{Re}(\phi)$$
$$\eta = \mathrm{Im}(\phi)$$

③ Express the PDE in the new coordinate system.

Note: Be careful about $b(x,y)$ (the factor of 2)

$\Rightarrow$ If you are unsure, note that if the PDE is written as

$$\alpha(x,y)u_{xx} + \beta(x,y)u_{xy} + \gamma(x,y)u_{yy} + \mathcal{L}^{(1)}(u) = g(x,y)$$

then $\dfrac{dy}{dx} = \dfrac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$

and this is entirely equivalent to the previous case (since $\beta = 2b$; $\alpha = a$, $\gamma = c$).

Review : Elements of Fourier Series

① Periodic function

- A periodic function is a function which satisfies the relation

$$f(x) = f(x+T) \quad \text{for all } x \text{, and a given } T > 0$$

T is the period of the function.

- Note that a function which is periodic with period T is also periodic with period nT for any $n \in \mathbb{N}$, $n > 0$. Usually T is the smallest real value for which $f(x) = f(x+T)$ holds.

② Orthogonality

- An inner product can be defined for any two functions on an interval $[a,b]$ as

$$\langle f, g \rangle = \int_a^b f(x) g(x) w(x)\, dx$$

where $w(x)$ is a fixed positive weight function (usually satisfying $\int_a^b w(x)dx = 1$.)

- Two functions are therefore orthogonal on $[a,b]$ provided $\langle f, g \rangle = 0$.

Property:

with $w(x) = \frac{1}{2L}$

- the functions $\sin\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{m\pi x}{L}\right)$ are orthogonal on $[-L, L]$ for all (m,n)
- the functions $\sin\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{m\pi x}{L}\right)$ are orthogonal on $[-L, L]$ for all $m \neq n$
- the functions $\cos\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{m\pi x}{L}\right)$ are orthogonal on $[-L, L]$ for all $m \neq n$

(3) Fourier Series

Any function f periodic with period $2L$ can be written as the series (called a Fourier Series)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

the Fourier coefficients
$$\begin{cases} a_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx \\ a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \end{cases}$$

<u>Proof</u>: Let $m>0$

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx$$

$$= \int_{-L}^{L}\left[a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]\cos\left(\frac{m\pi x}{L}\right)dx$$

$$= \int_{-L}^{L} a_0\cos\left(\frac{m\pi x}{L}\right)dx + \int_{-L}^{L}\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

$$+ \int_{-L}^{L}\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

given some prayer for convergence of the series

$$= \int_{-L}^{L} a_m\cos^2\left(\frac{m\pi x}{L}\right)dx = \frac{2L}{2}a_m = La_m$$

so $$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx$$ as required

(and similarly for the other terms)

(4) Properties of Fourier a_n, b_n coefficients

- if $f(x)$ is an even function $(f(x) = f(-x))$ then $b_n = 0 \quad \forall n$

- if $f(x)$ is an odd function $(f(x) = -f(-x))$ then $a_n = 0 \quad \forall n$

- The Fourier coeffs. of the sum of two functions f and g is equal to the sum of the Fourier coefficients

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$g(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)$$

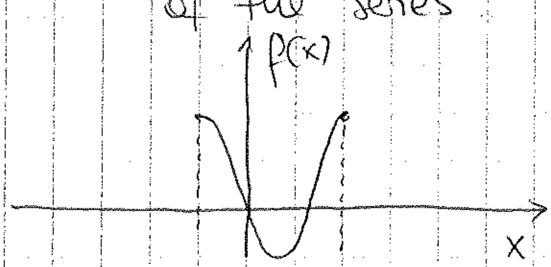
then

$$f+g\ (x) = (a_0 + A_0) + \sum_{n=1}^{\infty} (a_n + A_n) \cos\left(\frac{n\pi x}{L}\right) + (b_n + B_n) \sin\left(\frac{n\pi x}{L}\right)$$

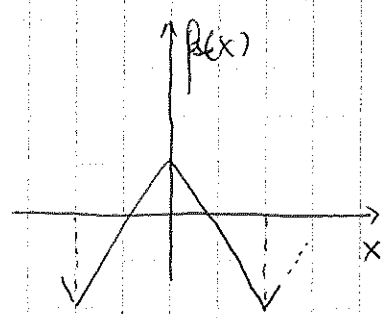
<u>BUT</u> : not true for the product!

- The Fourier series can be differentiated / integrated term by term to obtain the Fourier series of the derivative / integral of a function.

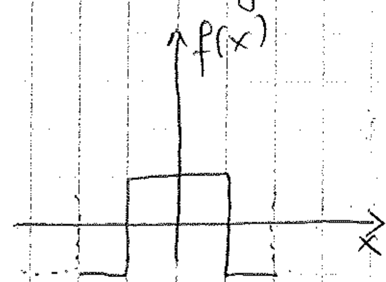
- The smoother the function, the quicker the convergence of the series:



$a_n, b_n \sim \frac{1}{n^3}$ or faster



$a_n, b_n \sim \frac{1}{n^2}$



$a_n, b_n \sim \frac{1}{n}$.