3.3.1 Canonical form of Hyperbolic equations

Consider a hyperbolic eq. $\mathcal{L}(u) = a u_{xx} + 2b u_{xy} + c u_{yy} + d u_x + e u_y + f = g$

To change it into its canonical form we require a coordinate transform $(x, y) \to (\xi, \eta)$ such that

$$A = C = 0 \qquad \text{(in the notation of 3.2)}$$

$$\Leftrightarrow \begin{cases} a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0 \\ a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0 \end{cases}$$

→ two equations are equivalent

Now rewrite $a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = a\left[\xi_x^2 + \frac{2b}{a}\xi_x\xi_y + \frac{c}{a}\xi_y^2\right]$

provided $a \neq 0$ $= a\left[\left(\xi_x + \frac{b}{a}\xi_y\right)^2 + \frac{c}{a}\xi_y^2 - \frac{b^2}{a^2}\xi_y^2\right]$

$$= a\left[\left(\xi_x + \frac{b}{a}\xi_y\right)^2 - \frac{b^2}{a^2}\xi_y^2\left(1 - \frac{c}{a}\frac{a^2}{b^2}\right)\right]$$

$$= a\left[\left(\xi_x + \frac{b}{a}\xi_y\left(1 + \sqrt{1 - \frac{ca}{b^2}}\right)\right) \cdot \left(\xi_x + \frac{b}{a}\xi_y\left(1 - \sqrt{1 - \frac{ca}{b^2}}\right)\right)\right]$$

So $a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0$ if and only if

$$\xi_x + \frac{b}{a}\left(1 + \sqrt{1 - \frac{ca}{b^2}}\right)\xi_y = 0$$

OR

$$\xi_x + \frac{b}{a}\left(1 - \sqrt{1 - \frac{ca}{b^2}}\right)\xi_y = 0.$$

→ Let's choose ξ a solution of $\xi_x + \frac{b}{a}\left(1 + \sqrt{1 - ca/b^2}\right)\xi_y = 0$

η —————— $\eta_x + \frac{b}{a}\left(1 - \sqrt{1 - ca/b^2}\right)\eta_y = 0$

- ξ is constant on the characteristics defined from

$$\frac{dx}{d\tau} = 1 \qquad \frac{dy}{d\tau} = \frac{b}{a}\left(1 + \sqrt{1 - ca/b^2}\right)$$

or $$dy/dx = \frac{b}{a}\left(1 + \sqrt{1 - ca/b^2}\right) = \frac{b + \sqrt{b^2 - ac}}{a}$$

and η is constant on characteristics satisfying

$$\frac{dy}{dx} = \frac{b}{a}\left(1 - \sqrt{1 - ac/b^2}\right) = \frac{b - \sqrt{b^2 - ac}}{a}$$

3.3.2 Examples

① The wave equation

$$u_{tt} - c^2 u_{xx} = 0 \qquad \delta(\mathcal{L}) = c^2 > 0$$

$\to$ a hyperbolic equation

To find a coordinate system (ξ, η) in which the wave equation is reduced to its canonical form, we must solve

$$\xi_t^2 - c^2 \xi_x^2 = 0$$

$$\Leftrightarrow \quad (\xi_t - c\xi_x)(\xi_t + c\xi_x) = 0$$

let ξ be the solution of $\xi_t - c\xi_x = 0$
η ——————————— $\eta_t + c\eta_x = 0$

ξ is constant on characteristics satisfying $\frac{dx}{dt} = -c$

$\Leftrightarrow \quad x = -ct + \text{constant}$

Let ξ be that constant, so that $\xi(x,t) = x + ct$

η is constant on characteristics satisfying $\frac{dx}{dt} = c$

$\Leftrightarrow \quad x = ct + \text{constant}$

Let η be that constant, so that $\eta(x,t) = x - ct$.

In the new coordinate system, we verify that

$$u_{tt} - c^2 u_{xx} = -u_{\xi\eta} \cdot 4c^2 = 0$$

Indeed $\quad u_t = cu_\xi - cu_\eta \qquad u_{tt} = c^2 u_{\xi\xi} + c^2 u_{\eta\eta} - 2c^2 u_{\eta\xi}$

$\quad u_x = u_\xi + u_\eta \qquad u_{xx} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\eta\xi}$

$$\Rightarrow \text{So} \quad -2c^2 u_{\eta\xi} - 2c^2 u_{\eta\xi} = 0 \Rightarrow \boxed{u_{\eta\xi} = 0.}$$

We can now find the solutions straightforwardly:

$$u_{\xi\eta} = 0 \iff u = F(\xi) + G(\eta)$$
$$= F(x+ct) + G(x-ct)$$

where F and G are chosen to satisfy the required boundary conditions.

SKIP (2) The Tricomi equation

$$u_{xx} + x u_{yy} = 0 \qquad \delta(\mathcal{L}) = -x$$

So the equation is hyperbolic for $x<0$.

→ We restrict the following work to the $x<0$ domain.

We seek the change of variable $(x,y) \to (\xi, \eta)$ which will simplify it into a canonical form

⇒ we require

$$\xi_x^2 + x\,\xi_y^2 = 0 = \xi_x^2 - |x|\,\xi_y^2$$

$$\iff \left(\xi_x + \sqrt{|x|}\,\xi_y\right)\left(\xi_x - \sqrt{|x|}\,\xi_y\right) = 0$$

let ξ be the solution of $\xi_x + \sqrt{|x|}\,\xi_y = 0$

→ ξ is constant on characteristics determined from

$$\frac{dy}{dx} = \sqrt{|x|}$$

$$\iff \quad y = \frac{2}{3}|x|^{\frac{3}{2}} + \text{constant}$$

so $\quad \xi = y - \frac{2}{3}|x|^{\frac{3}{2}}$

Similarly for η, $\quad \frac{dy}{dx} = -\sqrt{|x|}\quad$ so

$$y = -\frac{2}{3}|x|^{\frac{3}{2}} + \text{constant}$$

$$\Rightarrow \eta = y + \frac{2}{3}|x|^{\frac{3}{2}}$$

then $\xi_y = 1, \quad \xi_x = -|x|^{\frac{1}{2}} \qquad \xi_{yy} = 0 \quad \xi_{xx} = -\frac{1}{2|x|^{1/2}}$

$\eta_y = 1 \quad \eta_x = |x|^{1/2} \qquad \eta_{yy} = 0 \quad \eta_{xx} = \frac{1}{2|x|^{1/2}}$

so $u_x = \xi_x u_\xi + \eta_x u_\eta \qquad u_y = u_\xi + u_\eta$

$= -|x|^{1/2} u_\xi + |x|^{1/2} u_\eta$

$u_{xx} = -\frac{1}{2}|x|^{-\frac{1}{2}} u_\xi + \frac{1}{2}|x|^{-\frac{1}{2}} u_\eta - |x|^{\frac{1}{2}}\left[\xi_x u_{\xi\xi} + \eta_x u_{\xi\eta}\right]$

$+ |x|^{1/2}\left[\xi_x u_{\xi\eta} + \eta_x u_{\eta\eta}\right]$

$= |x|(u_{\xi\xi} + u_{\eta\eta}) - |x| u_{\xi\eta} + \frac{1}{2}|x|^{-\frac{1}{2}}(u_\eta - u_\xi)$

$u_{yy} = \left[\xi_y u_{\xi\xi} + \eta_y u_{\xi\eta} + \xi_y u_{\eta\xi} + \eta_y u_{\eta\eta}\right]$

$= u_{\xi\xi} + u_{\eta\eta} + 2u_{\eta\xi}$

so

$|x|(u_{\xi\xi} + u_{\eta\eta}) - |x| u_{\xi\eta} + \frac{1}{2}|x|^{-\frac{1}{2}}(u_\eta - u_\xi) - |x|(u_{\xi\xi} + u_{\eta\eta} + 2u_{\eta\xi})$

$= -3|x| u_{\xi\eta} + \frac{1}{2}|x|^{-\frac{1}{2}}(u_\eta - u_\xi) = 0$

$\Leftrightarrow \; -3|x|^{\frac{3}{2}} u_{\xi\eta} + \frac{1}{2}(u_\eta - u_\xi) = 0$

but $|x|^{3/2} = \frac{3}{4}(\eta - \xi)$

so finally $\quad u_{\xi\eta} + \frac{4}{9} \frac{1}{\xi - \eta} \cdot \frac{1}{2}(u_\eta - u_\xi) = 0$

is the canonical form of the Tricomi equation

3.3.2 Parabolic equations

To transform a parabolic equation into its canonical form, we require a change of coordinate acting such that

$$B = C = 0 \quad \text{(in the notation of 3.2)}$$

However since by definition $AC - B^2 = 0$, it is sufficient to require that $C = 0$

$\Rightarrow$ we need $\quad a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0$

But now recall that $ac - b^2 = 0$ so this is a perfect square so that it can be rewritten as

$$a\left(\eta_x + \sqrt{\frac{c}{a}}\,\eta_y\right)^2 = 0$$

alternatively $\quad \dfrac{1}{a}\left(a\eta_x + b\eta_y\right)^2 = 0.$

$\Rightarrow$ we can take η solution of the first order PDE

$$a\eta_x + b\eta_y = 0$$

$\Rightarrow$ η constant on the characteristics defined by $\dfrac{dy}{dx} = \dfrac{b}{a}$

[Note that this time ξ can be any function of x and y such that the Jacobian of (ξ,η) doesn't vanish]

Example: $\quad x^2u_{xx} - 2xy\,u_{yx} + y^2u_{yy} + xu_x + yu_y = 0$

$$\delta(\mathcal{L}) = x^2y^2 - x^2y^2 = 0$$

The characteristics satisfy $\dfrac{dy}{dx} = \dfrac{xy}{x^2} = -\dfrac{y}{x}$

So $\quad \ln y = -\ln x + \text{const}$

or $\quad y = \dfrac{K}{x}$ $\Rightarrow$ take $\eta = xy$ and for simplicity $\xi = x$

$$u_x = u_\xi + y u_\eta = v_\xi + \frac{\eta}{\xi} v_\eta \quad \text{since} \quad \xi_x = 1 \quad \xi_y = 0$$
$$u_y = x u_\eta = \xi v_\eta \qquad \eta_x = y \quad \eta_y = x$$

$$u_{xx} = u_{\xi\xi} + y u_{\xi\eta} + y^2 u_{\eta\eta} = v_{\xi\xi} + \frac{\eta}{\xi} v_{\eta\xi} + \left(\frac{\eta}{\xi}\right)^2 v_{\eta\eta}$$
$$u_{xy} = xy\, u_{\eta\eta} + x\, u_{\eta\xi} + u_\eta = \eta v_{\eta\eta} + \xi v_{\eta\xi} + v_\eta$$
$$u_{yy} = x^2 u_{\eta\eta} = \xi^2 v_{\eta\eta}$$

So we now have

$$\xi^2 \left[u_{\xi\xi} + \frac{2\eta}{\xi} u_{\eta\xi} + \frac{\eta^2}{\xi^2} u_{\eta\eta} \right]$$
$$-2\eta \left[\eta u_{\eta\eta} + \xi u_{\eta\xi} + u_\eta \right]$$
$$+ \frac{\eta^2}{\xi^2} \left[\xi^2 u_{\eta\eta} \right] + \xi \left(u_\xi + \frac{\eta}{\xi} u_\eta \right) + \frac{\eta}{\xi} \left(\xi u_\eta \right) = 0$$

$$\Leftrightarrow \quad \xi^2 u_{\xi\xi} + \xi u_\xi = 0 \quad \Leftrightarrow \quad \boxed{u_{\xi\xi} + \frac{1}{\xi} u_\xi = 0}$$

$\rightarrow$ the canonical form required.

This is now a simple ODE for $v = \frac{\partial u}{\partial \xi}$