CHAPTER 3 Second order linear PDEs : Canonical form

3.1 Definition

- A second order linear PDE has the general form

$$\mathcal{L}(u) = a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} \leftarrow \boxed{\text{principal part}}$$

$$+ \ d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y)$$

- The principal part is the part of the equation that only involves second-order derivatives.

- The discriminant of the linear operator $\mathcal{L}$ is defined as

$$\delta(\mathcal{L}) = b^2(x,y) - a(x,y)c(x,y)$$

and, in the general case, will be a function of x and y.

Examples

- the wave equation: $u_{tt} = c^2(x)u_{xx}$

write as $u_{tt} - c^2(x)u_{xx} = 0$

so $\delta(\mathcal{L}) = c^2(x)$

- the heat equation $u_t = ku_{xx}$

write as $ku_{xx} - u_t = 0$

so $\delta(\mathcal{L}) = -k \cdot 0 = 0$

• the Laplace equation: $u_{xx} + u_{yy} = 0$

$\rightarrow \; \delta(\mathcal{L}) = -1$

<u>Definition</u>: An operator is hyperbolic / parabolic / elliptic at a point (x, y) if $\delta(\mathcal{L})$ is respectively >0, $=0$ or <0 at this point

The operator for the wave equation is hyperbolic at all points (assume $c^2(x) > 0$).

______ the heat equation is parabolic at all points

______ Laplace equation is elliptic at all points

$\Rightarrow$ An equation is hyperbolic / parabolic / elliptic in a domain D if its corresponding operator is hyperbolic / parabolic / elliptic at all points in D

3.2 Properties of the discriminant under a change of coordinates

The sign of the discriminant of an operator $\mathcal{L}$ is invariant under a <u>change of coordinates</u> from (x, y) to (ξ, η) (such that the Jacobian $J = \xi_x \eta_y - \xi_y \eta_x \neq 0$ for all (x, y)).

In other words, the type of an equation is an intrinsic property of the equation and is independent of the coordinate system in which the equation is written.

Proof Let $\xi = \xi(x,y)$ $\Rightarrow$ now, $u(x,y) = w(\xi(x,y), \eta(x,y))$

$\eta = \eta(x,y)$

then $\frac{\partial u}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial w}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial w}{\partial \eta} \rightarrow \frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}$

$\frac{\partial u}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial w}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial w}{\partial \eta} \rightarrow \frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial \xi}{\partial x}\frac{\partial w}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial w}{\partial \eta}\right]$$

$$= \frac{\partial^2 \xi}{\partial x^2}\frac{\partial w}{\partial \xi} + \frac{\partial \xi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial \xi}\right) + \frac{\partial^2 \eta}{\partial x^2}\frac{\partial w}{\partial \eta} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial \eta}\right)$$

$$= \frac{\partial^2 \xi}{\partial x^2}\frac{\partial w}{\partial \xi} + \frac{\partial \xi}{\partial x}\left[\frac{\partial \xi}{\partial x}\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial \eta}{\partial x}\frac{\partial^2 w}{\partial \eta \partial \xi}\right]$$

$$+ \frac{\partial^2 \eta}{\partial x^2}\frac{\partial w}{\partial \eta} + \frac{\partial \eta}{\partial x}\left[\frac{\partial \xi}{\partial x}\frac{\partial^2 w}{\partial \xi \partial \eta} + \frac{\partial \eta}{\partial x}\frac{\partial^2 w}{\partial \eta^2}\right]$$

$$= \frac{\partial^2 \xi}{\partial x^2}\frac{\partial w}{\partial \xi} + \left(\frac{\partial \xi}{\partial x}\right)^2\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 \eta}{\partial x^2}\frac{\partial w}{\partial \eta} + \left(\frac{\partial \eta}{\partial x}\right)^2\frac{\partial^2 w}{\partial \eta^2} + 2\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x}\frac{\partial^2 u}{\partial \xi \partial \eta}$$

and similarly

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}\frac{\partial^2 w}{\partial \eta^2} + \left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 w}{\partial \xi \partial \eta}$$

$$+ \frac{\partial w}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial w}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \xi}{\partial y^2}\frac{\partial w}{\partial \xi} + \left(\frac{\partial \xi}{\partial y}\right)^2\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 \eta}{\partial y^2}\frac{\partial w}{\partial \eta} + \left(\frac{\partial \eta}{\partial y}\right)^2\frac{\partial^2 w}{\partial \eta^2}$$

$$+ 2\frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial y}\frac{\partial^2 w}{\partial \xi \partial \eta}$$

So the new PDE is

$$\tilde{\mathcal{L}}(w) = A w_{\xi\xi} + 2B w_{\xi\eta} + C w_{\eta\eta} + D w_{\xi} + E w_{\eta} + F w = G$$

with
$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2$$
$$B = a\eta_x\xi_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\eta_y\xi_y$$
$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2$$

Another way of writing this is

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix}$$
$$= J \begin{pmatrix} a & b \\ b & c \end{pmatrix} J^T$$

Now since $\delta(\mathcal{L}) = -\begin{vmatrix} a & b \\ b & c \end{vmatrix}$ then

$$\delta(\tilde{\mathcal{L}}) = -\begin{vmatrix} A & B \\ B & C \end{vmatrix} = +|J|\,\delta(\mathcal{L})\,|J^T| = |J||J^T|\,\delta(\mathcal{L}) = |J|^2\,\delta(\mathcal{L})$$

$\Rightarrow$ so provided $|J| \neq 0$, $\delta(\tilde{\mathcal{L}})$ has the same sign as $\delta(\mathcal{L})$, as required.

3.3 Canonical forms

We now consider three types of equations:
- hyperbolic equations ($\delta(\mathcal{L}) > 0$ everywhere)
- parabolic equations ($\delta(\mathcal{L}) = 0$ ———)
- and elliptic equations ($\delta(\mathcal{L}) < 0$ ———)

It is possible to find a coordinate transform $(x,y) \to (\xi, \eta)$ reducing these equations to their canonical forms such that

- ($\delta(\tilde{\mathcal{L}}) = 1/4$) hyperbolic equations become $u_{\xi\eta} + l_1(u) = g(\xi,\eta)$
- ($\delta(\tilde{\mathcal{L}}) = 0$) parabolic equations ——— $u_{\xi\xi} + l_1(u) = g(\xi,\eta)$
- ($\delta(\tilde{\mathcal{L}}) = -1$) elliptic equations ——— $u_{\xi\xi} + u_{\eta\eta} + l_1(u) = g(\xi,\eta)$

where $l_1(u)$ is a linear operator of first order