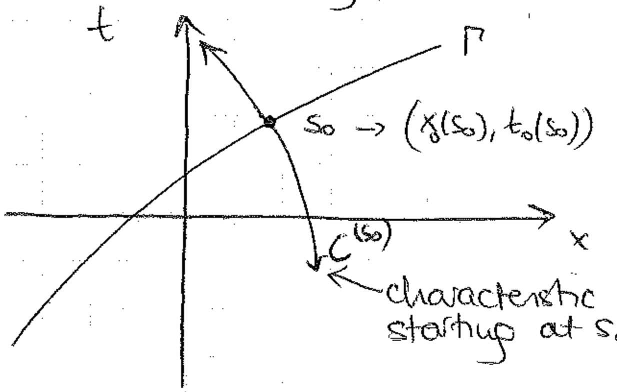
Step 2: Now that we have parametrized the "initial" condition curve we want to identify special curves along which the PDE behaves as an ODE. These are called characteristics.

① Suppose that for a selected point on the initial curve there exists only one characteristic emanating from it



⟹ let's parametrize this characteristic with the new parameter τ

$$C^{(s_0)}: \begin{cases} x^{(s_0)}(\tau) \\ t^{(s_0)}(\tau) \end{cases}$$

⟹ on this curve

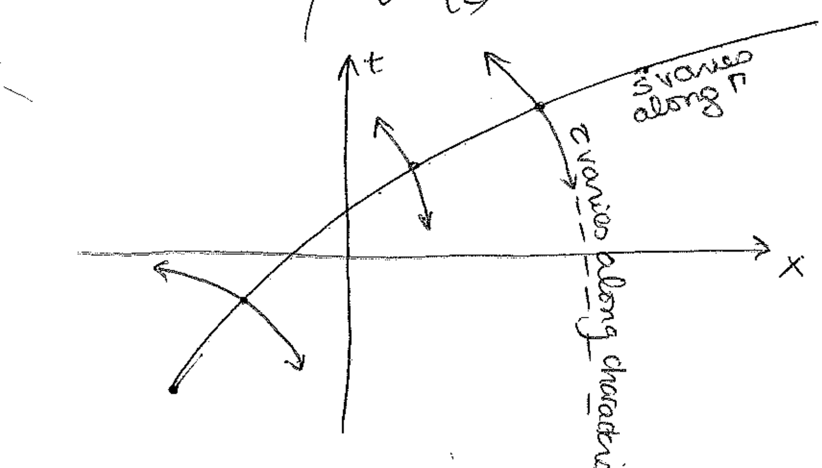
$$u\left(x^{(s_0)}(\tau), t^{(s_0)}(\tau)\right) = u^{(s_0)}(\tau)$$

u depends on τ only

② Now let's do this same construction for every point $[x_0(s), t_0(s)]$ on the initial condition curve

⟹ We get a family of characteristics, each starting from a point identified with the parameter s, and each parametrized with τ:

$$C^{(s)}: \begin{cases} x^{(s)}(\tau) \\ t^{(s)}(\tau) \end{cases} \quad \text{with } u^{(s)}(\tau) = u\left(x^{(s)}(\tau), t^{(s)}(\tau)\right)$$



$\Rightarrow$

$$C^{(s)} = \begin{cases} t^{(s)} = a\tau + \text{constant specific to this characteristic} \\ x^{(s)} = b\tau + \text{constant} \quad " \quad " \quad " \quad " \end{cases}$$

Suppose we require that when $\tau = 0$ we are on the initial condition curve then at $\tau = 0$

$$t^{(s)} = t_0(s)$$
$$x^{(s)} = x_0(s)$$

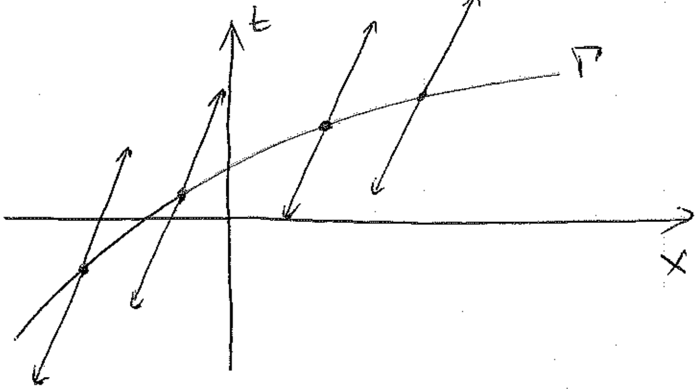
thus

$$C^{(s)} = \begin{cases} t^{(s)} = a\tau + t_0(s) \\ x^{(s)} = b\tau + x_0(s) \end{cases}$$

is the parametric equation for the characteristic emanating from the point $[x_0(s), t_0(s)]$ on the initial condition curve.

<u>What do these characteristics look like?</u>

Here, they are straight lines with slope $\frac{a}{b}$ in the (x,t) plane



<u>Note</u>: this is only true when the PDE has constant coefficients (see later)

<u>Step 3</u> <u>What is the solution to the PDE?</u>

Now we have to solve $\dfrac{\partial u}{\partial \tau} = c_1 u + c_0$ for each s

$$\Rightarrow \quad u = A(s)e^{c_1\tau} - \frac{c_0}{c_1} + \frac{c_0}{c_1}e^{c_1\tau} \qquad \text{(check this)}$$

where the arbitrary constant $A(s)$ is chosen such that $u = U_0(s)$ when $\tau = 0$:

$$u(s,\tau) = U_0(s)e^{c_1\tau} - \frac{c_0}{c_1} + \frac{c_0}{c_1}e^{c_1\tau}$$

$\Rightarrow$ Note that what we have really done, is to <u>remap</u> the (x,t) space onto the (s,τ) space

so that the function

$$u(x,t) \text{ is also } u\left(x^{(s)}(\tau), t^{(s)}(\tau)\right)$$
$$= u(s,\tau)$$

with the added requirement that the PDE

$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x}$ behaves like an ODE

when restricted to a characteristic $(s = \text{constant})$

<u>How do we impose this requirement?</u>

Note that $\left.\frac{\partial u}{\partial \tau}\right)_s$ is the derivative of u <u>along</u> a characteristic. (i.e holding s constant)

derivative of u w.r.t parameter τ at constant s

By multivariate chain rule, and using $\begin{pmatrix} x^{(s)}(\tau) \\ t^{(s)}(\tau) \end{pmatrix}$ on characteristic

$$\left.\frac{\partial u}{\partial \tau}\right)_s = \frac{\partial u}{\partial x}\left.\frac{\partial x}{\partial \tau}\right)_s + \frac{\partial u}{\partial t}\left.\frac{\partial t}{\partial \tau}\right)_s$$
$$= \frac{\partial u}{\partial x}\frac{d}{d\tau}\left[x^{(s)}\right] + \frac{\partial u}{\partial t}\frac{d}{d\tau}\left[t^{(s)}\right]$$

Going back to the original PDE, if

$$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = \frac{d}{d\tau}\left[t^{(s)}\right]\frac{\partial u}{\partial t} + \frac{d}{d\tau}\left[x^{(s)}\right]\frac{\partial u}{\partial x}$$

then $\Rightarrow \left.\frac{\partial u}{\partial \tau}\right)_s = c_1 u + c_0$

Now the PDE looks like an ODE for s held constant, i.e, along a characteristic.

This occurs when $\boxed{\frac{dt^{(s)}}{d\tau} = a \qquad \frac{dx^{(s)}}{d\tau} = b}$

How do we get a solution in terms of (x,t)?

Invert the system (if possible)

$$\begin{cases} t = a\tau + t_0(s) \\ x = b\tau + x_0(s) \end{cases}$$

to write τ and s in terms of x and t, then plug into $u(s,\tau)$

Example 1 Suppose we want to solve the simple transport equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + V_0 \dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = e^{-x^2/2} \end{cases} \qquad \text{(a Gaussian)}$$

Step 1: Parametrize the initial condition

$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = e^{-s^2/2} \end{cases}$$

Step 2 The characteristic curves are such that

$$\begin{pmatrix} a = 1 \\ b = V_0 \end{pmatrix} \qquad \begin{cases} \dfrac{\partial t^{(s)}}{\partial \tau} = 1 \\ \dfrac{\partial x^{(s)}}{\partial \tau} = V_0 \end{cases} \Rightarrow \begin{cases} t^{(s)} = \tau + t_0(s) \\ x^{(s)} = V_0\tau + x_0(s) \end{cases} \Rightarrow \begin{cases} t^{(s)} = \tau \\ x^{(s)} = V_0\tau + s \end{cases}$$

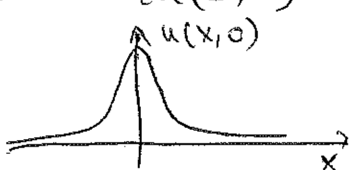
Step 3: The solution to $\dfrac{\partial u}{\partial \tau} = 0 \qquad (c_1 = c_0 = 0)$

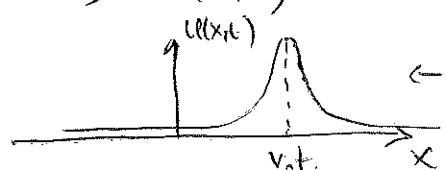
is $\quad u =$ constant on a characteristic

$$\Rightarrow u = u_0(s) = e^{-s^2/2}$$

Step 4: $\begin{cases} t = \tau \\ x = V_0\tau + s \end{cases} \Rightarrow \begin{cases} \tau = t \\ s = x - V_0 t \end{cases}$

so $\quad u(s,\tau) = e^{-s^2/2} \Rightarrow u(x,t) = e^{-(x-V_0t)^2/2}$



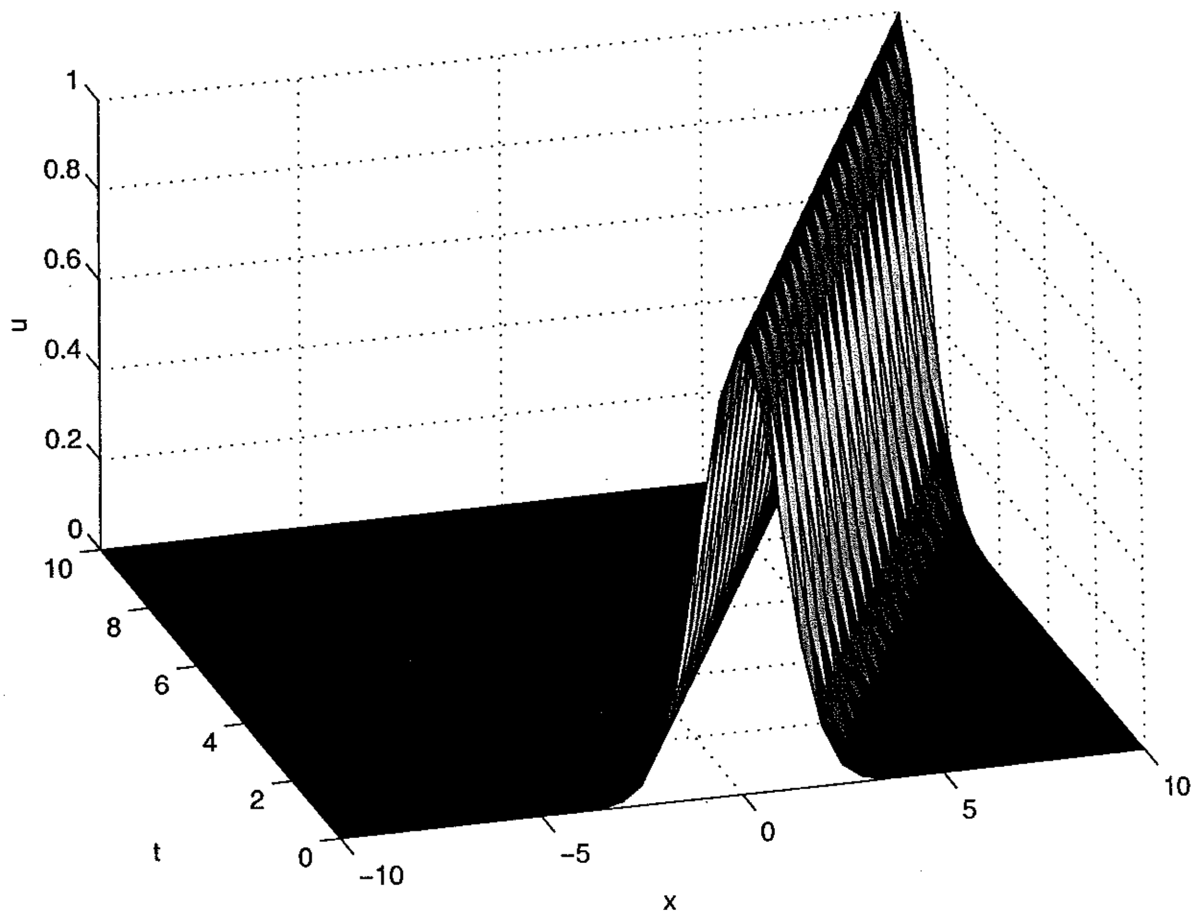


← a "travelling" Gaussian i.e. horizontally moved.

Visualization of the solution $u(x,t)$ to

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \\ u(x,0) = e^{-x^2/2} \end{cases}$$

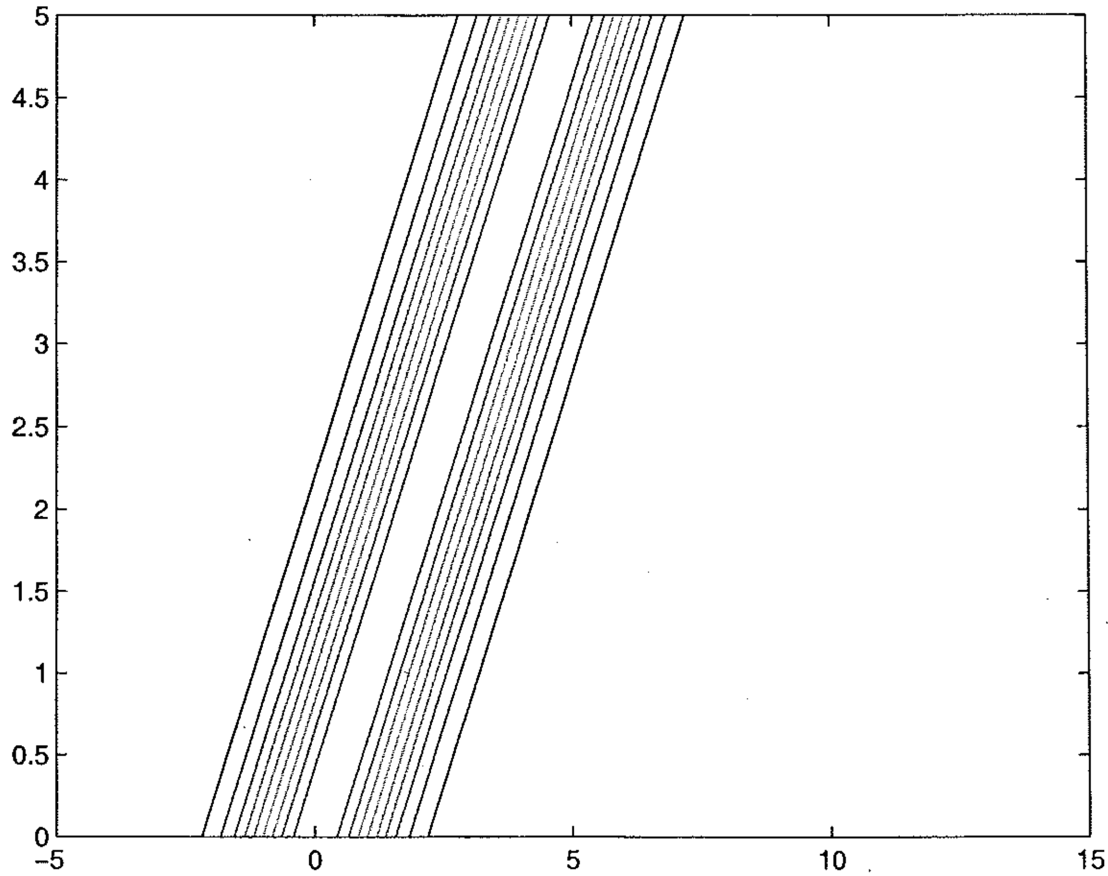
As a surface $u = u(x,t)$ in the 3-D space (x,t,u)



Contour levels of $u(x,t)$ solution of

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \\ u(x,0) = e^{-x^2/2} \end{cases}$$

The contours are also characteristics of the equation since $\frac{\partial u}{\partial \tau}$ is null on the characteristic.



Step 5 Always check your answer

$$\frac{\partial u}{\partial t} = -v_0 \, (x-v_0t)\, e^{-(x-v_0t)^2/2}$$

$$\frac{\partial u}{\partial x} = (x-v_0t)\, e^{-\frac{(x-v_0t)^2}{2}}$$

$$\frac{\partial u}{\partial t} + v_0 \frac{\partial u}{\partial x} = 0 \quad \checkmark$$

Example 2 : General case

$$\begin{cases} \dfrac{\partial u}{\partial t} + v_0 \dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = F(x) \end{cases}$$

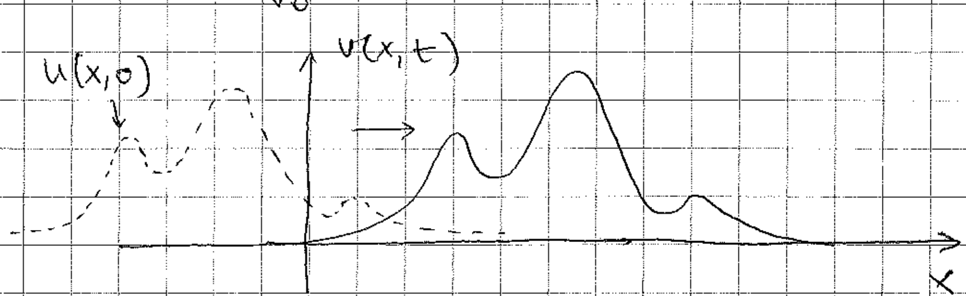
Step 1 :
$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = F(s) \end{cases}$$

Step 2 :
$$\begin{cases} \dfrac{dt}{d\tau} = 1 \\ \dfrac{dx}{d\tau} = v_0 \end{cases} \Rightarrow \begin{cases} t = \tau \\ x = v_0\tau + s \end{cases}$$

Step 3
$$\frac{du}{d\tau} = 0 \Rightarrow u = u_0(s) = F(s)$$

Step 4 :
$$\begin{cases} \tau = t \\ s = x - v_0 t \end{cases} \rightarrow \boxed{u(x,t) = F(x-v_0t)}$$

$\Rightarrow$ The initial condition $F(x)$ "moves" with velocity v_0



Example 3:
$$\begin{cases} \frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0 \\ u(x,0) = e^{-x^2} \end{cases}$$

Step 1:
$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = e^{-s^2} \end{cases}$$

Step 2
$$\begin{cases} \frac{dt}{d\tau} = 1 \\ \frac{dx}{d\tau} = x \end{cases} \Rightarrow \begin{cases} t = \tau + t_0(s) = \tau \\ x = x_0(s)e^{\tau} = se^{\tau} \end{cases}$$

Step 3
$$\frac{du}{d\tau} = 0 \Rightarrow u = u_0(s) = e^{-s^2}$$

Step 4:
$$\begin{cases} \tau = t \\ s = xe^{-t} \end{cases} \Rightarrow u(\cancel{x},t) = e^{-(xe^{-t})^2} = e^{-\frac{x^2}{e^{2t}}}.$$

This describes a Gaussian with constant amplitude 1, constant mean, but with a width which grows exponentially in time.

2.3.3 Semilinear equations

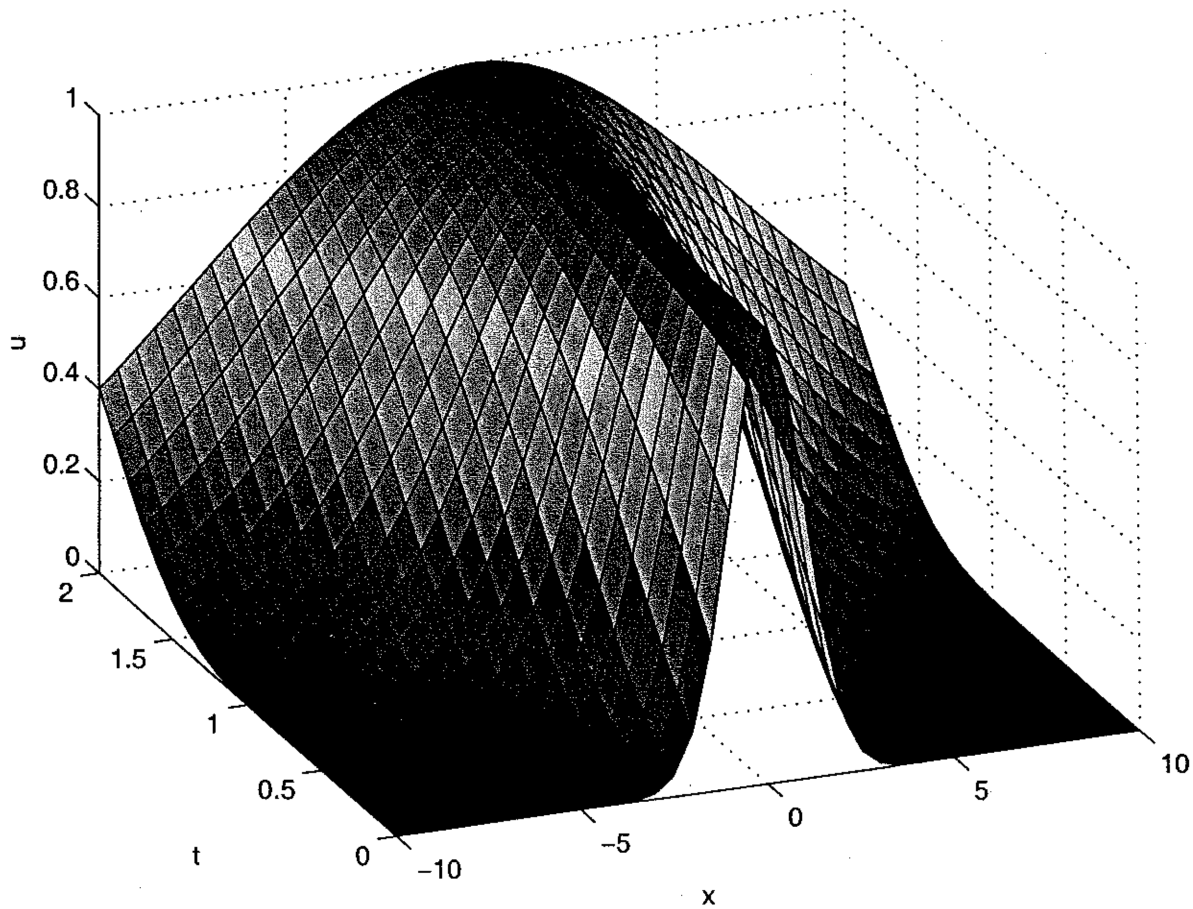
The method for semilinear equations:
$$\begin{cases} a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t,u) \\ u(x,0) = \phi(x) \end{cases}$$

is the same as for linear equations. However, note that the resulting ODE for u will be nonlinear.

Visualization of the solution $u(x,t)$ to

$$\begin{cases} \frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0 \\ u(x,0) = e^{-x^2/2} \end{cases}$$

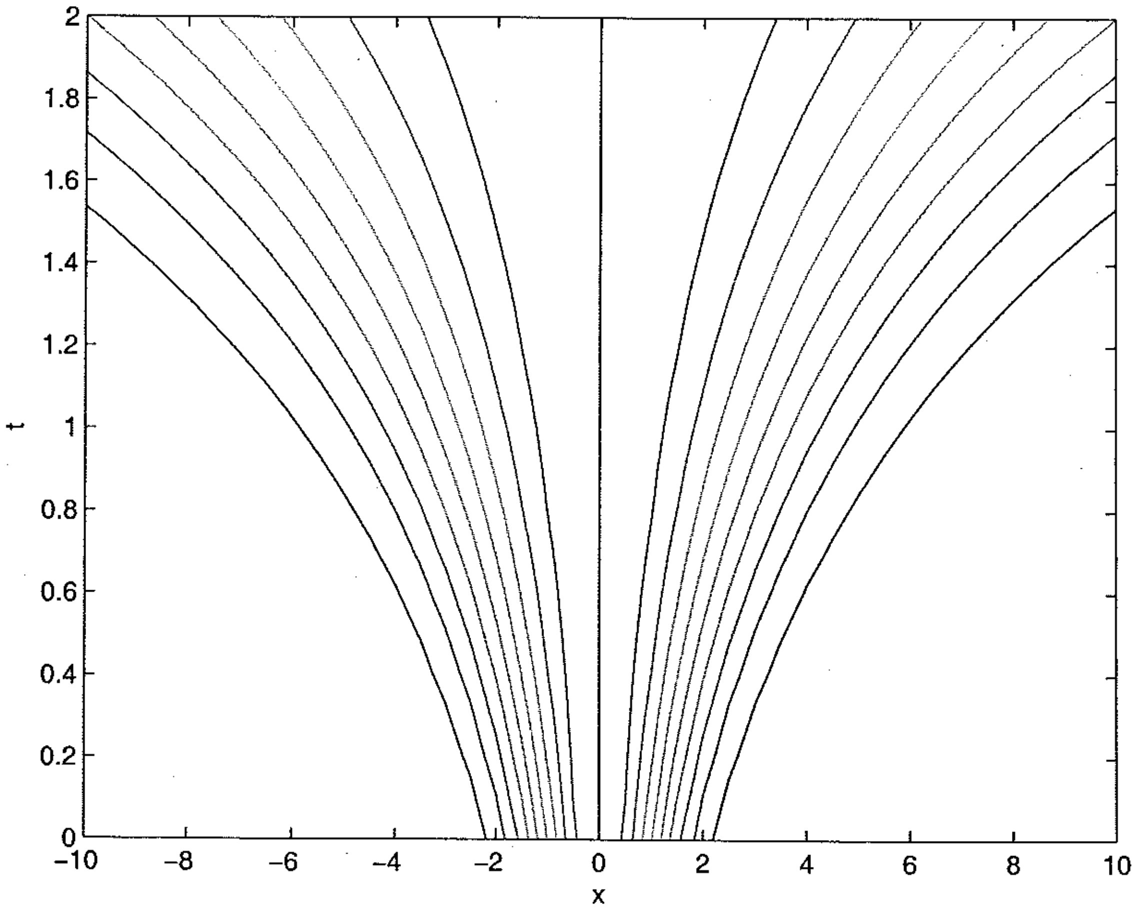
as a surface



Contour levels of the solution $u(x,t)$

to
$$\begin{cases} \dfrac{\partial u}{\partial t} + x\dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = e^{-x^2/2} \end{cases}$$

The contours are also characteristics of the equation since $\frac{\partial u}{\partial \tau}$ is null on a characteristic



Example
$$\begin{cases} \dfrac{\partial u}{\partial t} + x\dfrac{\partial u}{\partial x} = -e^{-u} \\ u(x,0) = e^{-x^2} \end{cases}$$

Step 1
$$\Gamma \quad \begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = e^{-s^2} \end{cases}$$

Step 2
$$\begin{cases} \dfrac{dt}{d\tau} = 1 \\ \dfrac{dx}{d\tau} = x \end{cases} \Rightarrow \begin{cases} t = \tau + t_0(s) = \tau \\ x = x_0(s)e^{\tau} = se^{\tau} \end{cases}$$

Step 3
$$\frac{du}{d\tau} = -e^{-u} \Rightarrow \frac{du}{e^{-u}} = -d\tau \Rightarrow e^{u}du = -d\tau$$

$$e^{u} = -\tau + k(s) \Rightarrow u = \ln(\tau + k(s))$$

$$\text{At} \quad \tau = 0 \quad e^{u_0(s)} = k(s) \Rightarrow k(s) = e^{e^{-s^2}}$$

$$u = \ln\left(-\tau + e^{e^{-s^2}}\right)$$

Step 4
$$\begin{cases} \tau = t \\ s = xe^{-t} \end{cases} \Rightarrow u(x,t) = \ln\left(-t + e^{e^{-x^2e^{-2t}}}\right)$$

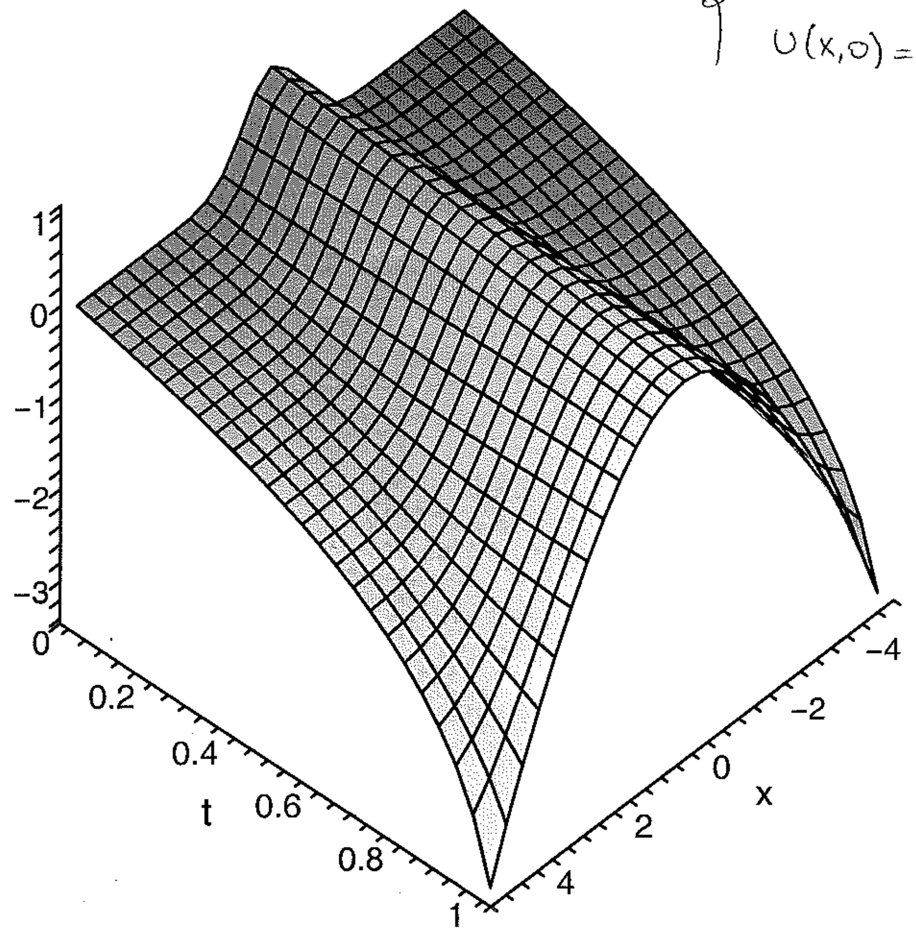
Step 5 Check:
$$\frac{\partial u}{\partial t} = \frac{-1 + 2x^2e^{-2t}e^{-x^2e^{-2t}}e^{e^{-x^2e^{-2t}}}}{-t + e^{e^{-x^2e^{-2t}}}}$$

$$\frac{\partial u}{\partial x} = -\frac{2xe^{-2t}e^{-x^2e^{-2t}}e^{e^{-x^2e^{-2t}}}}{-t + e^{e^{-x^2e^{-2t}}}}$$

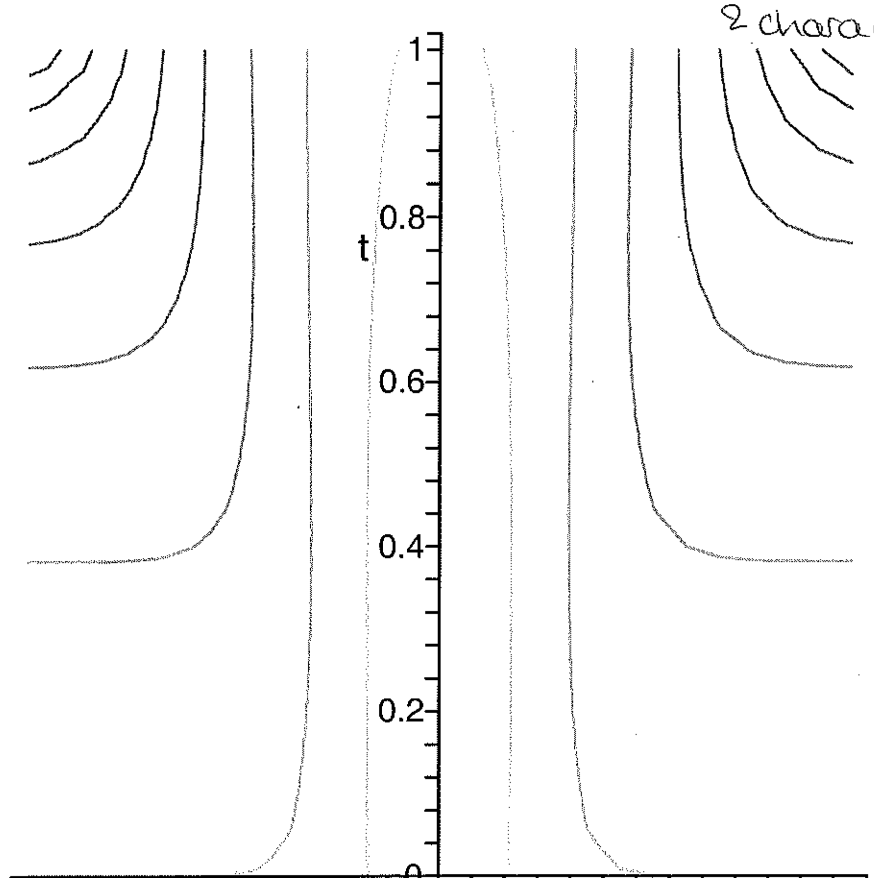
so ✓.

Solution 2 Contour plot for

$$\begin{cases} u_t + x u_x = -e^{-u} \\ u(x,0) = e^{-x^2} \end{cases}$$



Note that contour lines 2 characteristics do <u>not</u> coincide.



What did we learn?

(1) Method of solution of semilinear, first order PDES

Step 1: Parametrize the initial condition curve

Step 2: If $a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t,u)$

then the characteristics are found by solving the system

$$\begin{cases} \frac{\partial t^{(s)}}{\partial \tau} = a(x,t) \\ \frac{\partial x^{(s)}}{\partial \tau} = b(x,t) \end{cases}$$

Note that these are coupled ODES.

with the initial condition

$$t^{(s)}(\tau=0) = t_0(s)$$
$$x^{(s)}(\tau=0) = x_0(s)$$

Step 3: The solution to the PDE in (s,τ) is found by solving

$$\frac{\partial u^{(s)}}{\partial \tau} = c(x,t,u)$$

(note that x and t depend on s and τ)

Step 4: If possible, invert the system

$$\begin{cases} t(s,\tau) \\ x(s,\tau) \end{cases} \text{ to get } \begin{cases} \tau(x,t) \\ s(x,t) \end{cases}$$

and plug into $u(s,\tau)$ to get $u(x,t)$.

Step 5 Check answer.

② Note:

- When the linear PDE is homogeneous ($c(x,t) = 0$) then

$$\frac{\partial u}{\partial \tau} = 0 \Rightarrow$$ u is constant along characteristics; in other words, the characteristics are contour levels of the solution $u(x,t)$.

- When the PDE is not homogeneous then u is not constant along characteristics. The characteristics propagate the initial condition according to the equation

$$\frac{\partial u^{(s)}}{\partial \tau} = c\left(x^{(s)}(\tau), t^{(s)}(\tau); u^{(s)}(\tau)\right)$$

(see examples later)

③ Question

- What is the condition for the mapping

$$\begin{cases} x(s,\tau) \\ t(s,\tau) \end{cases}$$ to be invertible?

- What happens if the characteristics are somewhere parallel to the initial condition curve?