III Green's functions revisited

6.2.1 Fundamental solution of Laplace equation and applications

We saw that the function

$$\Gamma(x,y) = -\frac{1}{4\pi}\ln(x^2+y^2) = -\frac{1}{2\pi}\ln\sqrt{x^2+y^2}$$

is a solution of $\nabla^2\Gamma = 0$ everywhere in the plane (with no bcs) except at (x,y) where it is undefined.

<u>Definition</u>: $\Gamma(x,y;\xi,\eta) = -\frac{1}{4\pi}\ln\left((x-\xi)^2+(y-\eta)^2\right)$

is the fundamental solution of Laplace equation with a <u>pole</u> at (ξ,η)

6.2.2 Properties of Γ

Consider a domain D, and a function u solution of $\nabla^2 u = f$ in D, f smooth.

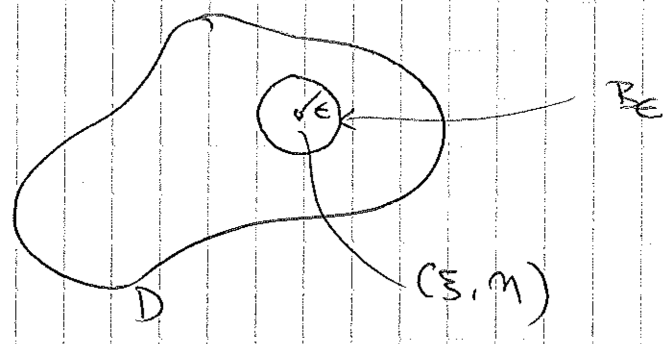
Then for all (ξ,η) in D then

$$u(\xi,\eta) = \int_{\partial D}\left[\Gamma(x,y;\xi,\eta)\partial_n u - u\partial_n\Gamma(x,y;\xi,\eta)\right]dS$$
$$-\int_D \Gamma(x,y;\xi,\eta)f(x,y)\,dV$$

<u>Proof</u>: Use Green's #2 identity with u and Γ in the following domain:

$$D_\epsilon = D - B_\epsilon$$

where B_ϵ is a "sphere" of radius ϵ centred on (ξ, η)



then

$$\int_{D_\epsilon} \left(\Gamma \nabla^2 u - u \nabla^2 \Gamma\right) dV = \int_{\partial D_\epsilon} \left(\Gamma \partial_n u - u \partial_n \Gamma\right) dS.$$

in D_ϵ, $\nabla^2 \Gamma = 0$ everywhere.

$\partial D_\epsilon = \partial D - \partial B_\epsilon$ so

$$\int_{\partial D_\epsilon} \Gamma \partial_n u - u \partial_n \Gamma \, dS = \int_{\partial D} \left(\Gamma \partial_n u - u \partial_n \Gamma\right) dS - \int_{\partial B_\epsilon} \left(\Gamma \partial_n u - u \partial_n \Gamma\right) dS.$$

$$\int_{\partial B_\epsilon} \Gamma \partial_n u \, dS = \int_{\partial B_\epsilon} -\frac{1}{2\pi} \ln \epsilon \, \partial_n u \, dS = -\frac{1}{2\pi} \ln \epsilon \int_{\partial B_\epsilon} \partial_n u \, dS$$

$$= -\frac{1}{2\pi} \ln \epsilon \int_{B_\epsilon} \nabla^2 u \, dV$$

$$= -\frac{1}{2\pi} \ln \epsilon \int_{B_\epsilon} f \, dV = O\left(\epsilon^d \ln \epsilon\right)$$

where d is the spatial dimension

so $\int_{\partial B_\epsilon} \Gamma \partial_n u \, dS \to 0$ when $\epsilon \to 0$.

$$\int_{\partial B_\epsilon} u \partial_n \Gamma \, dS = \int_{\partial B_\epsilon} -\frac{u}{2\pi\epsilon} \, dS = -\int_0^{2\pi} \frac{u(\xi + \epsilon\cos\theta, \eta + \epsilon\sin\theta)}{2\pi\epsilon} \, \epsilon \, d\theta$$

since $\Gamma = -\frac{1}{2\pi} \ln(r - r_0)$

So $$\int_{\partial B_\epsilon} u\, \partial_n \Gamma \, dS = -u(\xi,\eta) \quad \text{as } \epsilon \to 0$$

so finally, taking the limit as $\epsilon \to 0$ of all terms we get

$$\int_D \Gamma \nabla^2 u \, dV = \int_{\partial D} \Gamma \partial_n u - u\, \partial_n \Gamma \, dS - u(\xi,\eta) \qquad \square.$$

Corrolary: For any domain D, $\displaystyle\int_{\partial D} \partial_n \Gamma \, dS = -1$ (use $u=1$).

6.2.3 Unbounded domains

Let's first consider the case of an un-bounded domain, with the required condition that
$u(x,y) \to 0$ as $|x^2+y^2| \to +\infty$.

In that case, the surface term vanishes and we have

$$u(\xi,\eta) = -\int_{\mathbb{R}^2} \Gamma \nabla^2 u \, dV = -\int_{\mathbb{R}^2} \nabla^2 \Gamma \; u(x,y)\,dx\,dy$$

$\Rightarrow$ But Γ is such that $\nabla^2 \Gamma = 0$ everywhere except at (ξ,η)

$\Rightarrow \nabla^2 \Gamma$ is a δ- function

$\Rightarrow$ more precisely $\boxed{\nabla^2 \Gamma = -\delta(x-\xi,\, y-\eta)}$

conclusions: Γ is the solution of $\nabla^2 \Gamma = -\delta(x-\xi, y-\eta)$

$u(\xi,\eta) = -\displaystyle\int_{\mathbb{R}^2} \Gamma(x,y;\xi,\eta)\, f(x,y)\,dx\,dy$ is the solution of $\nabla^2 u = f$.

Γ is (-) the Greens function in the plane

Examples of applications

- The gravitational potential created by a distribution of mass $\rho(\underline{r})$ is found by solving the Poisson equation

$$\nabla^2 \phi = 4\pi G \rho(\underline{r})$$

- In an unbounded domain, we require that $|\phi(\underline{r})| \to 0$ as $|\underline{r}| \to \infty$.

$\Rightarrow$ we can apply the previously derived formula!

① In a 2D plane, we know that

$$\Gamma(x,y) = -\frac{1}{4\pi} \ln(x^2+y^2)$$

so

$$\phi(x,y) = -\int_{\mathbb{R}^2} dx'dy' \, 4\pi G \rho(x',y') \Gamma(x-x', y-y')$$

$$= + \int_{\mathbb{R}^2} dx'\,dy' \; G\rho(x',y') \ln\left((x-x')^2 + (y-y')^2\right)$$

This may not look familiar, but it is the formula for a 2D gravitational potential!

② This is generalizable to higher-dimensions: eg: 3D

Idea: Γ is the center-symmetric solution of the Laplacian operator: in 3D, seek $\Gamma = \Gamma(r)$ only

$$\Rightarrow \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Gamma}{\partial r}\right) = 0$$

$$\Rightarrow \quad r^2 \frac{\partial \Gamma}{\partial r} = K \quad \Rightarrow \quad \Gamma = -\frac{K}{r}$$

Again, we select a normalization such that

$$\Gamma(r) = + \frac{1}{4\pi r}$$

to guarantee, as in previous section, $\int_{\partial B_\epsilon} u \partial_n \Gamma \, dS = -u(\xi, \eta)$ (see above).

In cartesian coordinates: $\Gamma(x,y,z) = \dfrac{1}{4\pi \sqrt{x^2+y^2+z^2}}$

$\Rightarrow$ Now we know that in 3D

$$\phi(x,y,z) = -\int_{\mathbb{R}^3} \frac{dx'\,dy'\,dz' \; G\rho(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$\phi(\underline{r}) = -\int \frac{d^3\underline{r}' \; G\rho(\underline{r}')}{|r-r'|}$$

↑ the standard formula for the 3D gravitational potential ϕ generated by a distribution of mass $\rho(\underline{r})$.

6.2.4 Integral solution for the Dirichlet problem for the Poisson equation

Consider the Dirichlet Poisson problem $\begin{cases} \nabla^2 u = f & (x,y) \in D \\ u = g & (x,y) \in \partial D \end{cases}$

Given the expression

$$u(\xi, \eta) = \int_{\partial D} (\Gamma \partial_n u - u \partial_n \Gamma) \, ds - \int_D \Gamma \nabla^2 u \, dV$$

we know u on ∂D $(u = g)$
$\nabla^2 u$ in D so $(\nabla^2 u = f)$.

$$u(\xi, \eta) = \int_{\partial D} \Gamma \partial_n u - g u \, ds - \int \Gamma f \, dV$$

Problem: we do not know $\partial_n u$, so we cannot use this directly.

Idea: Consider the function

$$G(x, y; \xi, \eta) = \Gamma(x, y; \xi, \eta) - h(x, y; \xi, \eta)$$

where $h(x, y; \xi, \eta)$ is the solution to

$$\begin{cases} \nabla^2 h = 0 & \text{in } D \\ h = \Gamma & \text{on } \partial D \end{cases}$$

then $\nabla^2 G = -\delta(x - \xi, y - \eta)$ by construction and
$G = 0$ on ∂D so that using
Green's # 2 identity we now have

$$\int_D u \nabla^2 G - G \nabla^2 u \, dx dy = \int_{\partial D} (u \partial_n G - G \partial_n u) \, ds$$

$$(\Rightarrow) \quad u(\xi, \eta) = \left. \begin{aligned} & -\int_D G(x, y; \xi, \eta) f(x, y) \, dx dy \\ & - \int_{\partial D} g(x, y) \partial_n G \, ds \end{aligned} \right\} \begin{array}{l} \text{an integral} \\ \text{representation} \\ \text{of the solution} \\ \text{using Green's} \\ \text{function } G \end{array}$$

Note: This doesn't quite solve the problem yet since we still have to find h. However, there are tricks to find h for simple geometries (see later).

6.2.5 Integral solution of the Poisson equation for the Neumann problem

Consider
$$\begin{cases} \nabla^2 u = f & (x,y) \in D \\ \partial_n u = g & (x,y) \in \partial D \end{cases}$$

in the expression
$$u(\xi,\eta) = \int_{\partial D} \Gamma \partial_n u - u \partial_n \Gamma \, dS - \int_D \Gamma \nabla^2 u \, dV$$

we know $\partial_n u$ and $\nabla^2 u$ but not u on ∂D.

This time consider the function
$$N(x,y;\xi,\eta) = \Gamma(x,y;\xi,\eta) - h(x,y;\xi,\eta)$$

with $\begin{cases} \nabla^2 h = 0 \\ \partial_n h = \partial_n \Gamma + \frac{1}{L} \end{cases}$ where L is the length of ∂D.

Again $\nabla^2 N = -\delta(x,y;\xi,\eta)$ by construction

and $\partial_n N = \partial_n \Gamma - \partial_n \Gamma - \frac{1}{L} = -\frac{1}{L}$

Note: For $\nabla^2 h = 0$ and $\partial_n h = \partial_n \Gamma + \frac{1}{L}$ to have a solution we must verify that

$\int_{\partial D} \partial_n h \, dS = 0$. This is true because

$\int_{\partial D} \partial_n \Gamma \, dS = -1$ and $\int_{\partial D} \frac{1}{L} \, dS = 1$. $(*)$

Using Green's #2 identity we get

$$\int u\nabla^2 N - N\nabla^2 u \; dV = \int_{\partial D} u\,\partial_n N - N\partial_n u \; ds$$

$$-u(\xi,\eta) = \int_V N\nabla^2 u - \int_{\partial D} Ng \; ds - \int_{\partial D} \frac{u}{L}\, ds$$

$$\boxed{u(\xi,\eta) = -\int_V Nf + \int_{\partial D} Ng \; ds + \int_{\partial D} \frac{u}{L}\, ds}$$

Note: N is defined to within an additive constant $\int_{\partial D} \frac{u}{L} ds$

(*): to prove $\int_{\partial D} \partial_n \Gamma \, ds = -1$ simply use

$$u(\xi,\eta) = \int_{\partial D} \Gamma\partial_n u - u\,\partial_n\Gamma - \int_D \Gamma\nabla^2 u \, ds$$

with $u \equiv 1$ (since it holds for any smooth function u)

then $1 = \int_{\partial D} 0 - \partial_n\Gamma \; ds - 0$.

Alternatively recall that $\nabla^2\Gamma$ is a δ function so that

$$\int_{\partial D} \partial_n\Gamma \; ds = \int_D \nabla^2\Gamma \; dV = 1$$

Note: In any Neumann problem the solution is defined to within an additive constant. The term

$\int_{\partial D} \frac{u}{L} ds$ incorporates that constant.

(For instance, choose $\int_{\partial D} u\, ds = 0$ and $\int_{\partial D} N(x,y,\xi,\eta)\, ds = 0$ as normalisations, then the solution is uniquely defined).

6-3 Examples

(1) Consider the Dirichlet problem in the half-space

$$\nabla^2 \psi = f(x,y) \qquad (x>0)$$

$$\psi = 0 \quad \text{for} \quad x=0$$

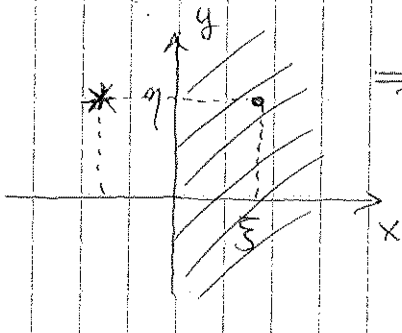
The fundamental solution to Laplace equation is $\Gamma(x,y;\xi,\eta) = -\frac{1}{2\pi} \ln\left((x-\xi)^2+(y-\eta)^2\right)$

We now need to find solutions to

$$\begin{cases} \nabla^2 h = 0 & \text{for } x>0 \\ h = \Gamma & \text{on } x=0 \end{cases} \quad (\text{so } h(0,y;\xi,\eta) = \Gamma(0,y;\xi,\eta))$$

Method of images (reflection principle)

Idea: Construct the function h to be the symmetric function of Γ across the domain boundary:



$\Rightarrow$ If Γ has a pole in (ξ,η), construct a function that has a pole in $(-\xi,\eta)$

here $h = \Gamma(x,y;-\xi,\eta)$

so that $G = \Gamma(x,y;\xi,\eta) - \Gamma(x,y;-\xi,\eta)$

We can verify that

$\nabla^2 G = \nabla^2 \Gamma$ in D (since $\nabla^2 h = 0$ everywhere in D (the pole is outside of D))

$G(x=0) = 0$

since $h(0,y,\xi,\eta) = \Gamma(0,y;\xi,\eta)$

so the solution to the problem is

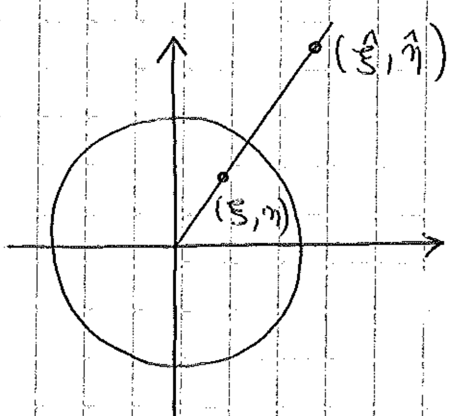
$$u(\xi,\eta) = -\int_{x>0} G(x,y,\xi,\eta)\, f(x,y)\, dx\,dy - \int_{x=0} 0 \cdot \partial_n G \, dS = -\int_{y=-\infty}^{+\infty}\int_{x=0}^{\infty} G(x,y,\xi,\eta)\, f(x,y)\, dx\,dy$$

(2) A "similar" construction can be used to compute h in a disk.

Consider the Dirichlet problem in the disk

$$\begin{cases} \nabla^2 \psi = f(x,y) & (x^2+y^2)^{1/2} < R \\ \psi(x,y) = g(x,y) & (x^2+y^2)^{1/2} = R \end{cases}$$

$\Rightarrow$ we want to find, for each (ξ,η), the function $h(x,y;\xi,\eta)$ such that



$$\begin{cases} \nabla^2 h = 0 & \text{in the disk} \\ h(x,y;\xi,\eta) = \Gamma(x,y;\xi,\eta) & \text{on the disk} \end{cases}$$

Trick: Consider the "inverse" point of (ξ,η), $(\hat\xi,\hat\eta)$ defined as

$$\begin{pmatrix} \hat\xi \\ \hat\eta \end{pmatrix} = \frac{R^2}{\xi^2+\eta^2} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

and let

$$h(x,y,\xi,\eta) = \Gamma\left[\frac{\sqrt{\xi^2+\eta^2}}{R}\, x, \frac{\sqrt{\xi^2+\eta^2}}{R}\, y; \frac{\sqrt{\xi^2+\eta^2}}{R}\, \hat\xi, \frac{\sqrt{\xi^2+\eta^2}}{R}\, \hat\eta \right]$$

$$= \Gamma\left[\frac{\sqrt{\xi^2+\eta^2}}{R}\, x, \frac{\sqrt{\xi^2+\eta^2}}{R}\, y; \frac{R}{\sqrt{\xi^2+\eta^2}}\, \xi, \frac{R}{\sqrt{\xi^2+\eta^2}}\, \eta \right]$$

Check:

(1) $\nabla^2 h = h_{xx} + h_{yy} = \frac{\xi^2+\eta^2}{R^2} \nabla^2 \Gamma = 0$

(2) $h(x,y;\xi,\eta) = -\frac{1}{2\pi} \ln\left[\frac{\xi^2+\eta^2}{R^2}(x-\hat{\xi})^2 + \frac{\xi^2+\eta^2}{R^2}(y-\hat{\eta})^2 \right]$

should be equal to

$\Gamma(x,y;\xi,\eta) = -\frac{1}{2\pi} \ln\left[(x-\xi)^2 + (y-\eta)^2 \right]$

on the circle $x^2+y^2 = R^2$

$$\frac{\xi^2+\eta^2}{R^2}\left[(x-\hat{\xi})^2 + (y-\hat{\eta})^2 \right]$$

$$= \frac{\xi^2+\eta^2}{R^2}\left[x^2+y^2 - 2x\hat{\xi} - 2y\hat{\eta} + \hat{\xi}^2 + \hat{\eta}^2 \right]$$

$$= (\xi^2+\eta^2) - (2x\xi + 2y\eta) + \frac{R^2}{\xi^2+\eta^2}(\xi^2+\eta^2)$$

$$= R^2 - 2x\xi + 2y\eta + (\xi^2+\eta^2)$$

$$= (x-\xi)^2 + (y-\eta)^2 \qquad \square.$$

$\Rightarrow$ The Green's function on the disk of radius R is

$$G(x,y;\xi,\eta) = \Gamma(x,y;\xi,\eta) - \Gamma\left(\frac{\sqrt{\xi^2+\eta^2}}{R}x, \frac{\sqrt{\xi^2+\eta^2}}{R}y; \frac{R}{\sqrt{\xi^2+\eta^2}}\xi, \frac{R}{\sqrt{\xi^2+\eta^2}}\eta \right)$$

$$= -\frac{1}{2\pi} \ln\left((x-\xi)^2 + (y-\eta)^2 \right)$$

$$+ \frac{1}{2\pi} \ln\left[\left(\frac{\sqrt{\xi^2+\eta^2}}{R}x - \frac{R}{\sqrt{\xi^2+\eta^2}}\xi \right)^2 + \left(\frac{\sqrt{\xi^2+\eta^2}}{R}y - \frac{R}{\sqrt{\xi^2+\eta^2}}\eta \right)^2 \right]$$

and the solution to the Dirichlet problem

$$\begin{cases} \nabla^2 u = f & \text{in } D \\ u = g & \text{on } D \end{cases} \quad \text{is} \quad u(\xi,\eta) = -\iint_D G f \, dx\,dy - \int_{\partial D} g \frac{\partial G}{\partial r} \, d\ell$$

(r = radial coordinate)