(4) Fundamental 1st order PDE. the transport equation

$$\boxed{\frac{\partial f}{\partial t} + \nabla \cdot (\underline{u} f) = 0}$$

where $\underline{u}$ is a velocity field (known)

Expression in Cartesian coordinates:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(u f) + \frac{\partial}{\partial y}(v f) + \frac{\partial}{\partial z}(w f) = 0$$

if

$$\underline{u} = (u, v, w)$$

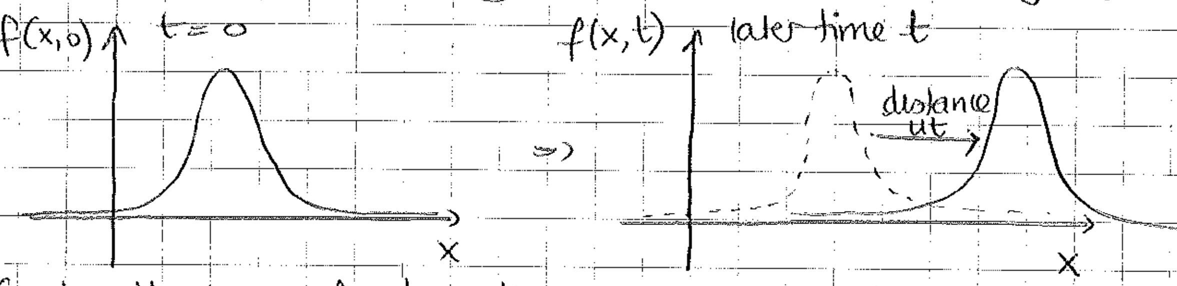
- This equation expresses the conservation of the quantity f through transport by the velocity field $\underline{u}$.

- If $\underline{u}$ is a constant field then

$$\boxed{\frac{\partial f}{\partial t} + \underline{u} \cdot \nabla f = 0}$$

(note that this is also generally true for all vector fields $\underline{u}$ s.t. $\nabla \cdot u = 0$)

A solution is merely "moved around" by $\underline{u}$.



(note the similarity with propagation of wave).

See ppt for movies in more than 1D.

(5) Additional conditions & well-posedness (a first look at)

- When modelling physical problems, the PDE is <u>always</u> accompanied by additional conditions, usually in the form of
  - initial conditions (for a time-dependent problem)
  - boundary conditions (for a problem on a finite domain)
  - regularity conditions (either, regularity/bound at infinity, or regularity at a coordinate singularity)

The behaviour of a solution depends as much of the PDE than on these additional conditions

- For a given PDE with given additional conditions, there can be no, one or many possible solutions

<u>example</u>. $u_t = u_x$

- Given this PDE <u>without</u> any additional conditions, there are an infinity of solutions ($u = c$ for all $c \in \mathbb{R}$)
- Given this PDE together with $u(x, t=0) = \phi(x)$ there is a unique solution (see Chapter 2).
- Given this PDE with $u(x, t=-x) = \phi(x)$ then there is no solution (see Chapter 2) or an infinite # of solutions

- For a given equation and set of additional conditions, a small change in the parameters of the equation or of the conditions can lead to a <u>big</u> change in the solution.

Example $\quad u_t = -u_{xx} \quad t>0$

$u(x,0) = 1$

→ obvious solution is $u(x,t) = 1$

but if we had chosen $u(x,0) = 1 + \frac{1}{n} \cdot \sin(nx)$

then the solution is

$$u(x,t) = 1 + \frac{1}{n} e^{n^2 t} \sin(nx) \qquad \text{(CHECK THIS)}$$

Now for n large enough $1 + \frac{1}{n}\sin(nx) \simeq 1$

but after a time t large enough $\frac{e^{n^2 t}}{n} \gg 1$

so a small difference in initial conditions creates an enormous difference in the final solution.

Definition:

a PDE (or set of PDE) and its associated additional conditions is a well-posed problem if

- it has a solution
- this solution is unique
- the structure of the solution is unchanged by infinitesimal variations of the parameters and/or of the additional conditions.

Typically: • a well-thought, well-modelled physical problem will result in a well-posed problem. because in nature, the solution exists.

(although simplifying assumptions & shortcuts often lead to ill-posed problems)

BUT: not necessarily always the case. For nonlinear PDEs:

- multiple solutions can exist in which slight changes in the boundary conditions lead to one, or the other equilibrium
- sometimes, it is the discontinuous solutions that interest us (shock physics)

⇒ called weak solutions

CHAPTER 2 First order PDEs (in 2 dimensions)

2.1 General formulae

A first order PDE in 2 dimensions is in the form of

$$\mathcal{F}\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) = 0$$

- A first order <u>linear</u> PDE in 2 dimensions is

$$a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t)u + d(x,t).$$

NONLINEAR PDES:

- A first order <u>semilinear</u> PDE in 2D is

$$a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t,u)$$

- A first order quasilinear PDE is

$$a(x,t,u)\frac{\partial u}{\partial t} + b(x,t,u)\frac{\partial u}{\partial x} = c(x,t,u).$$

A fully nonlinear first order PDE is none of the above!

2.2 Method of characteristics for quasilinear equations

2.2.1 Warmup example

Let's study $u_t = c_0 u + G(x,t)$ <u>c_0 constant</u>

Note that for each x, it is actually an ODE in t

$\rightarrow$ fix x and solve it!

Use integrating factor method (for example)

$$u_t - c_0 u = G(x,t)$$

$\Rightarrow$ We try to find an integrating factor $\mu(x,t)$ such that

$$\mu u_t - \mu c_0 u = \mu G(x,t)$$

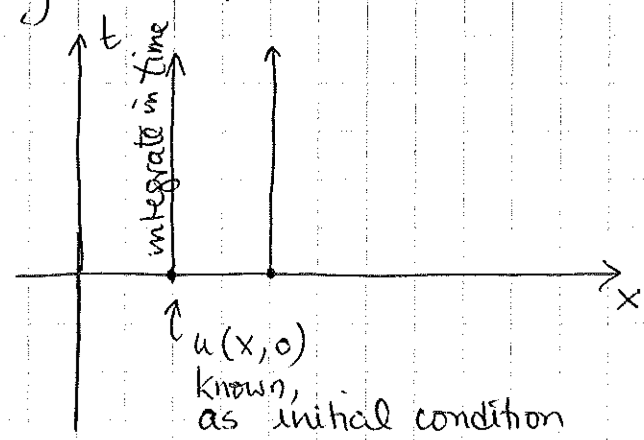
$$= \frac{\partial}{\partial t}(\mu u)$$

$\longrightarrow$ take $\mu = e^{-c_0 t}$ so

$$\frac{\partial}{\partial t}\left(e^{-c_0 t} u\right) = e^{-c_0 t} G(x,t)$$

$$e^{-c_0 t} u(x,t) - e^{-c_0 \cdot 0} u(x,0) = \int_{t'=0}^{t'=t} e^{-c_0 t'} G(x,t')\,dt'$$

Again, this can be done for each value of x separately: we are solving the equation by integrating along lines of constant x



<u>Initial conditions</u> (u is known at $t=0$)

Suppose we require that $u(x,0) = 3x$ then

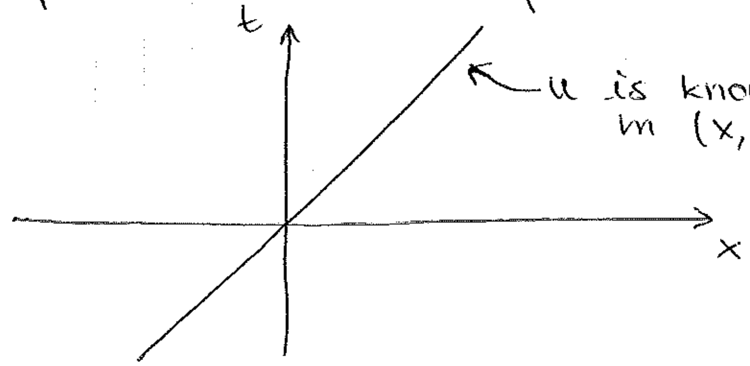
$$u(x,t) = e^{+c_0 t} u(x,0) + \int_{t'=0}^{t'=t} e^{-c_0(t'-t)} G(x,t')\,dt'$$

$$= 3x e^{+c_0 t} + \int_{t'=0}^{t'=t} e^{-c_0(t'-t)} G(x,t')\,dt'$$

$\rightarrow$ a unique solution.

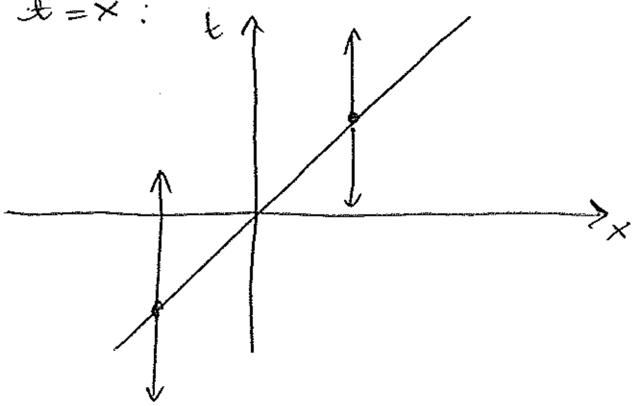
Other kinds of additional condition

(1) Suppose instead we require that $u(x,x) = 3x$



← u is known on this line in (x,t) plane

Then, instead of integrating from $t'=0$, we integrate from $t'=x$:



Mathematically:

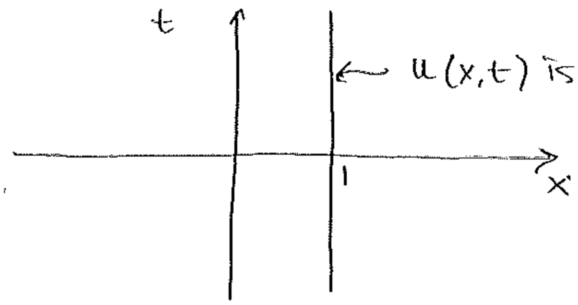
$$e^{-C_0 t} u(x,t) - e^{-C_0 x} u(x,x) = \int_{t'=x}^{t'=t} e^{-C_0 t'} G(x,t')\, dt'$$

$$\Rightarrow \quad e^{-C_0 t} u(x,t) = e^{-C_0 x} \cdot 3x + \int_x^t e^{-C_0 t'} G(x,t')\, dt'$$

$$u(x,t) = e^{-C_0 (x-t)} \cdot 3x + \int_x^t e^{-C_0 (t'-t)} G(x,t')\, dt'$$

→ again, there is a unique solution to the PDE with the given additional condition.

(2) Now suppose we set $C_1 = 0$ and try to impose as additional condition $u(1,t) = 2t$



Problem! The additional condition doesn't satisfy the equation

$$\frac{\partial u}{\partial t} = 2 \quad \rightarrow \quad u_t - C_0 u = 2 - 2C_0 t \neq 0 .$$

$\rightarrow$ there are no solutions to the equation!

(3) Now suppose $u(1,t) = 2e^{C_0 t}$ then

$$u_t - C_0 u = 2C_0 e^{C_0 t} - 2C_0 e^{C_0 t} = 0 \quad \checkmark$$

$\Rightarrow$ the additional condition satisfies the equation

But note that any function of the form
$$u(x,t) = f(x)\, e^{C_0 t}$$
satisfies the PDE and the additional condition provided $f(1) = 2$

$\Rightarrow$ there are an ∞ of solutions to the problem!

Conclusion: • Depending on the additional conditions chosen, there can be one, no or an ∞ of solutions to the problem. Case (1) is well-posed while cases (2) and (3) are ill-posed

• What is the difference between cases (1), (2) and (3)?

Note that in case (1), the additional condition crosses all lines of constant x, while in cases (2) and (3), the additional condition is a line of constant x.

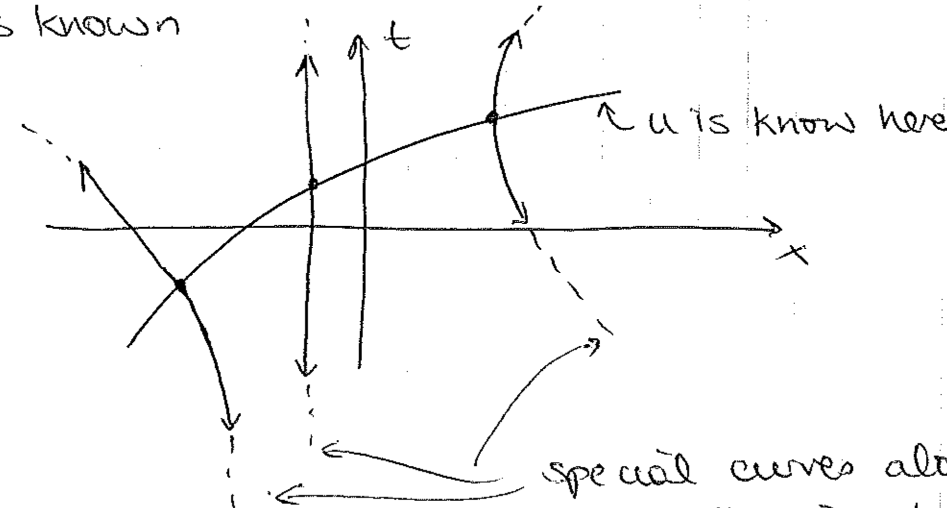
2.2.2 Gearing up to the general method

Now consider the linear transport equation with constant coefficients.

$$a \frac{\partial u}{\partial t} + b \frac{\partial u}{\partial x} = C_1 u + C_0$$

where a, b, C_1, C_0 are constants.

<u>Idea</u>: we would like to find curves (as before) along which we could integrate the PDE <u>as if</u> it were an ODE, from an initial or additional condition line where $u(x,t)$ is known



DETOUR: Review of parametric curves

Any curve in $\mathbb{R}^n$ can be represented by a set of parametric equations

$$\begin{cases} x_1 = f_1(s) \\ x_2 = f_2(s) \\ \vdots \\ x_n = f_n(s) \end{cases}$$

where s is the parameter.

<u>Examples</u>: A circle in $\mathbb{R}^2$ (x,y) centred on $(0,0)$ has the equation $\begin{cases} x = R\cos(s) \\ y = R\sin(s) \end{cases}$ where R is the radius

- A straight line in $\mathbb{R}^2$ has the parametric equation

$$\begin{cases} x = as + c \\ y = bs + d \end{cases}$$

<u>Check</u> : eliminate s to get

$$y = b\left(\frac{x-c}{a}\right) + d = \frac{b}{a}x + \left(d - \frac{bc}{a}\right)$$

<u>Property of parametric curves</u>

The tangent vector to the curve $\{f_1(s) \dots f_n(s)\}$ is

$$\underline{df} = \begin{pmatrix} df_1/ds \\ df_2/ds \\ \vdots \\ df_n/ds \end{pmatrix}$$

<u>Examples</u>: • the tangent vector to the line

$$\begin{cases} x = as + c \\ y = bs + d \end{cases} \quad \text{is} \quad \underline{df} = \begin{pmatrix} dx/ds \\ dy/ds \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

• Suppose you are travelling from SC to Big Sur. Your <u>trajectory</u> is given by the parametric curve

$$\begin{pmatrix} x(t) \\ y(t) \\ h(t) \end{pmatrix} \begin{matrix} \leftarrow \text{latitudinal position } x \\ \leftarrow \text{longitudinal position } y \\ \leftarrow \text{height} \end{matrix}$$

Your <u>velocity</u> is the tangent vector to the trajectory

$$\underline{v} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dh}{dt} \end{pmatrix} \begin{matrix} \leftarrow \text{North-South velocity} \\ \leftarrow \text{East-West velocity} \\ \leftarrow \text{vertical velocity.} \end{matrix}$$

<u>Note</u>: A parametrization is <u>NOT</u> unique.

<u>Example</u>: $\begin{cases} x = R\sin s \\ y = R\cos s \end{cases}$ and $\begin{cases} x = R\sin(s^2) \\ y = R\cos(s^2) \end{cases}$ represent the <u>same</u> curve

Back to the first order PDE

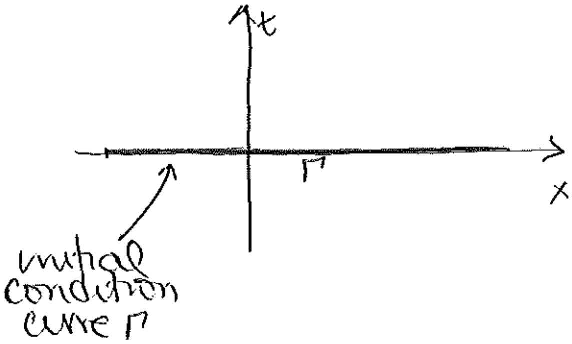
Step 1: We represent the additional condition curve as a parametric curve with parameter s

Suppose we know $u(x,t)$ on a particular curve Γ in the (x,t) plane. Let's parametrize Γ with the functions $x_0(s)$, $t_0(s)$ such that

$$\Gamma = \begin{cases} x_0(s) \\ t_0(s) \end{cases}$$

then on this curve $u(x_0(s), t_0(s)) = u_0(s)$

Examples: • Suppose we want to impose
$$u(x,0) = 3x$$



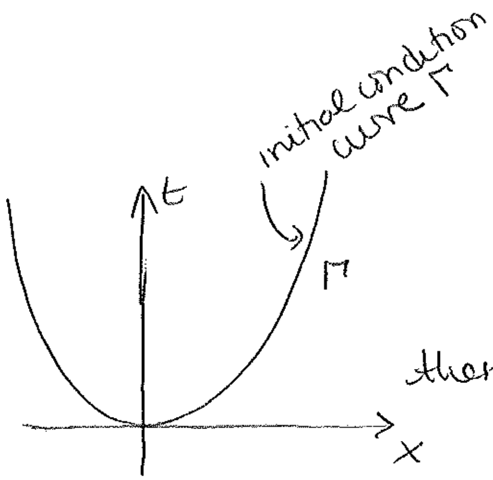
The initial condition curve has $t=0$ for all x $\Rightarrow$

Parametrize it (for example) as

$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \end{cases} \Rightarrow u_0(s) = 3s$$

or we could also use

$$\begin{cases} x_0(s) = s^2 \\ t_0(s) = 0 \end{cases} \Rightarrow u_0(s) = 3s^2$$



• Suppose $u(x, x^2) = e^x$

then

$$\begin{cases} x_0(s) = s \\ t_0(s) = s^2 \end{cases} \Rightarrow u_0(s) = e^s$$

Note: Since there are many possible parametric representations of the same curve, always try to choose the simplest one:

Prefer
$$\begin{cases} x_0(s) = s \\ t_0(s) = s^2 \end{cases} \quad \text{over} \quad \begin{cases} x_0(s) = \ln(s^2) \\ t_0(s) = [\ln(s^2)]^2 \end{cases} \quad !$$