Review Dirac Delta function $\delta(x)$

① **Definition**. The Dirac Delta function $\delta(x)$ is defined as having the following properties

① $$\int_{-\infty}^{+\infty} \delta(x)\,dx = 1$$

(Note that ② implies ①)

② $$\int_{-\infty}^{+\infty} \delta(x) f(x)\,dx = f(0) \quad \text{for any function } f(x)$$

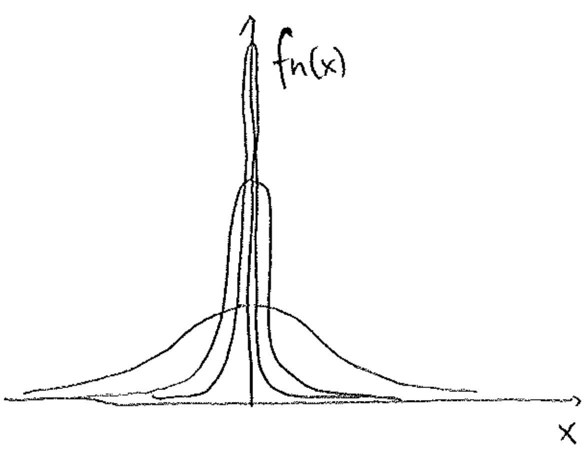
③ $$\delta(x) = 0 \qquad \forall x \neq 0 .$$

In fact, $\delta(x)$ is not properly speaking a function

It can be considered as the limit of the sequence of functions

$$f_n(x) = \frac{n}{\sqrt{2\pi}} e^{-\frac{n^2x^2}{2}} \qquad n \geq 1, \quad n \to \infty .$$

for example



Note: It can also be considered as a limit of many other functions:

$$\lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2}$$

$$\lim_{\epsilon \to 0} \frac{1}{\pi x} \sin\left(\frac{x}{\epsilon}\right)$$

② Fourier transform

The Fourier transform of the Dirac Delta function is

$$D(\omega) = \int_{-\infty}^{+\infty} e^{i\omega x} \delta(x)\,dx = 1$$

$\rightarrow$ the FT of the Dirac delta is constant.

By inverse transform we also see that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} D(\omega)e^{-i\omega x}\, d\omega = \delta(x)$$

$$\Rightarrow \quad \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega x}\, d\omega = \delta(x)$$

Note that if we define

$$f_n(x) = \frac{1}{2\pi}\int_{-n}^{n} e^{-i\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-n}^{n} \cos\omega x + i\sin\omega x \, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{1}{\omega}\sin\omega x - \frac{i}{\omega}\cos\omega x\right]_{-n}^{n}$$

$$= \frac{1}{2\pi}\left[\frac{2}{n}\sin nx\right] = \frac{1}{n\pi}\sin nx$$

$$= \frac{\epsilon}{\pi}\sin\left(\frac{x}{\epsilon}\right) \quad \text{if } \epsilon = \frac{1}{n}$$

$\rightarrow$ We recover the idea of

$$\delta(x) = \lim_{n\to\infty} f_n(x) = \lim_{\epsilon\to 0} \frac{\epsilon}{\pi}\sin\left(\frac{x}{\epsilon}\right).$$

II Elliptic equations. Green's functions

I) Prototype elliptic problems

- Recall that elliptic equations can always be cast into their canonical form

$$u_{xx} + u_{yy} + \mathcal{L}^{(1)}(u) = 0$$

where $\mathcal{L}^{(1)}(u)$ is a linear operator acting on u.

- Therefore, the simplest equations considered are
  - the Laplace equation
  $$u_{xx} + u_{yy} \left(= \nabla^2 u\right) = 0$$
  - the Poisson equation
  $$u_{xx} + u_{yy} = F(x,y)$$

I.1 Divergence theorem & consequences for elliptic problems

- Recall: the divergence theorem states that

$$\int_V \nabla \cdot \underset{\sim}{u}\, dV = \int_{\partial V} \underset{\sim}{u} \cdot \underline{n}\, dS$$

$\underline{n}$ = normal unit vector to the "surface" ∂V.

V can be here a volume/surface, then ∂V is a surface/contour.

- $\nabla^2 u = \nabla \cdot (\nabla u)$

cf. in cartesian: $\nabla u = \begin{pmatrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial u/\partial z \end{pmatrix}$

so $$\nabla \cdot \nabla u = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)$$
$$= \nabla^2 u.$$

As a result, for an elliptic Poisson-type PDE we see that

$$\int_V \nabla^2 u \, dV = \int_{\partial V} \nabla u \cdot \underline{n} \, dS = \int_V F(\underline{r}) \, dV.$$

$\Rightarrow$ the nature of the equation provides a constraint on the solution at the boundaries.

<u>Consequence</u>: recall that for Neumann-type boundary conditions we impose $\nabla u \cdot \underline{n}$ $(= \partial_n u)$ on the domain contour

$\Rightarrow$ only when $\displaystyle\int_{\partial V} \partial_n u \cdot dS = \int_V F(\underline{r}) dV$ (*)

will the Neumann problem have a solution.

<u>In general</u>:
- the problem of existence of solutions for elliptic equations is much more complex than for parabolic / hyperbolic equations.
- Provided the domain considered is <u>bounded</u> and <u>smooth enough</u>.

see later on

- Solutions to the Dirichlet problem exist and are unique
- solutions to the Neumann problem exist (if * holds) but are not unique ($u = v + K$, $K \in \mathbb{R}$ is also solution)
- solutions to the Robin problem exist and are unique.

A.2 Harmonic functions

Definition: a harmonic function $u(x,y)$ in a domain D is a function satisfying
$$\nabla^2 u = 0 \quad \text{for all } (x,y) \in D$$

Examples: (among others).

① Let's look for polynomial solutions:

e.g. 2nd order: (quadratic form)
$$ax^2 + bxy + cy^2 = 0$$
$$\Rightarrow \quad 2a + 2c = 0$$
$\Rightarrow$ • $a = -c$
• b can be any value

So any function of the kind
$$A(x^2 - y^2) + Bxy + \alpha x + \beta y + \gamma = 0$$
is a harmonic function in the whole plane.

② Let's look for center symmetric solutions

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0$$
$$\Rightarrow \quad r\frac{\partial u}{\partial r} = K \quad \Rightarrow \quad \frac{\partial u}{\partial r} = \frac{K}{r} \Rightarrow$$
$$u = +K\ln r + K'$$

Note that $u(r)$ is <u>not</u> defined at $r=0$.
$\rightarrow$ harmonic in $\mathbb{R}^2 - \{0,0\}$

In cartesians:
$$u(x,y) = \frac{K}{2}\ln(x^2+y^2) + K'.$$

1.3 The maximum principles and mean value principle

(1) <u>Theorem</u>: The weak maximum principle

let D be a <u>bounded</u> domain, and $u(x,y)$ a function continuous & differentiable in D satisfying
$\nabla^2 u = 0$ in D (a harmonic function)

Then the maximum of u is achieved <u>on</u> the boundary ∂D

(2) <u>Corrolary</u>: the above theorem also holds for the minimum of u, because

- $v = -u$ is also a harmonic function and
- $\max(v) = \min(u)$

Idea behind the theorem (see Textbook)

For any local maximum <u>within</u> D, we necessarily have $\nabla^2 u \leq 0$
(recall, for 1D functions, x_0 is a local maximum of u $\Rightarrow$ $u''(x_0) \leq 0$).

$\Rightarrow$ a function <u>without</u> local maxima within D satisfies $\nabla^2 v > 0$ (v can have maxima on ∂D)

So let's construct $v = u + \epsilon f(x,y)$
where $\nabla^2 f =$ constant (positive) and $\epsilon > 0$.
and $f \geq 0$
(for example, $f(x,y) = x^2 + y^2$)

then
- $\max(v)$ is on the boundary
- $\max(u) = \max(v - \epsilon f(x,y))$

let $\epsilon \to 0$ so $\max(u)$ must be on the boundary too.

(3) The mean value principle

Let D be a planar domain, let u be a harmonic function in D and (x_0, y_0) be a point within D.

Consider $R \in \mathbb{R}$ such that the disk D_R centred at (x_0, y_0) with radius R is fully contained in D. then

$$u(x_0, y_0) = \oint_{\partial D_R} u(x(s), y(s))\, dl \cdot \left[\oint_{\partial D_R} dl\right]^{-1}$$

= the average of u over the circle bounding D_R

<u>Note</u> here the parametrization of the circle is

$$x(s) = x_0 + R\cos(s) \rightarrow dx = -R\sin(s)\,ds$$
$$y(s) = y_0 + R\sin(s) \rightarrow dy = R\cos(s)\,ds$$

so $dl = \sqrt{dx^2 + dy^2} = R\,ds$ with $s \in [0, 2\pi]$ $\Rightarrow \left[\oint_{\partial D_R} dl\right]^{-1} = \frac{1}{2\pi R}$.

<u>Idea of Proof</u> Consider the function $\oint_{\partial D_r} u(x(s), y(s))\,dl \cdot \left[\oint_{\partial D_r} dl\right]^{-1} = F(r)$

then $F(r) = \frac{1}{2\pi \cancel{r}} \int_0^{2\pi} u(x_0 + r\cos(s), y_0 + r\sin(s))\,\cancel{r}\,ds$

$$\rightarrow \frac{\partial F}{\partial r} = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{\frac{\partial}{\partial r}\left(u(x_0 + r\cos(s), y_0 + r\sin(s))\right) ds}_{\oint \partial_n u \, dl}$$

$= 0$ since u is a harmonic function

$\Rightarrow F = \text{constant} = F(0) = u(x_0, y_0)$. so $F(R) = u(x_0, y_0)$

(4) Theorem (strong maximum principle)

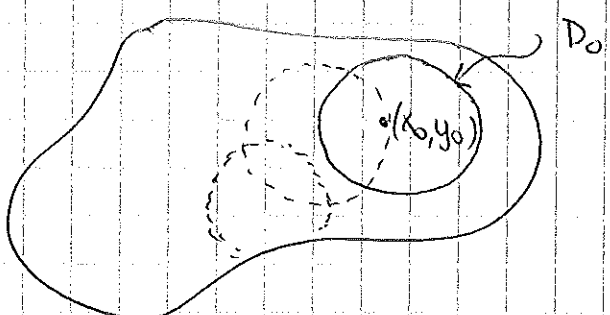
let u be a harmonic function in a domain D. If u attains its maximum (or minimum) in D then u is constant

Idea behind the proof

If u attains its maximum somewhere at (x_0, y_0) within D, then construct a disk D_0 around (x_0, y_0) contained in D

By the mean value theorem & the fact that (x_0, y_0) is a maximum, we deduce that u is equal to the max everywhere on the contour of D_0. Since the MVT is also true for all disks within D_0, we conclude that $u = u_{max}$ for all points in D_0.

Finish the proof by "paving" D with connected disks



9.4 Consequence: uniqueness of solutions in bounded domains for specific boundary conditions

Example for the Dirichlet problem

Consider the problem

$$\nabla^2 u = f(x,y) \quad \text{for } (x,y) \in D$$
$$u(x,y) = g(x,y) \quad \text{for } (x,y) \in \partial D$$

(where D is a bounded domain).

To prove uniqueness, consider two solutions v_1 and v_2 to the problem. Then

$v = v_1 - v_2$ is solution to $\begin{cases} \nabla^2 v = 0 & \text{in } D \\ v(x,y) = 0 & \text{on } \partial D \end{cases}$

Since v attains both minimum and maximum on ∂D then

$$0 \leq v(x,y) \leq 0 \qquad \forall\, (x,y) \in D$$

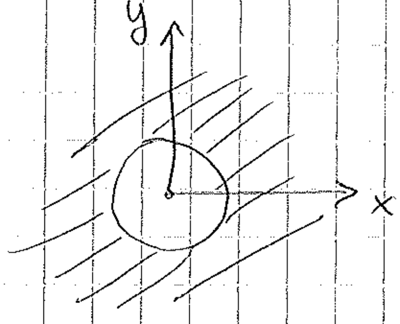
$\rightarrow$ v is identically 0 so $v_1 = v_2$

<u>Note</u>: Since the weak maximum principle only holds for bounded domains, the uniqueness of solutions to the Dirichlet problem only holds for bounded domains

<u>Counter example</u>

Consider $\nabla^2 u = 0 \quad : x^2+y^2 > 4$

$u(x,y) = 1 \qquad x^2+y^2 = 4$



then $u(x,y) = 1$ is a solution

$$u(x,y) = \frac{\ln(x^2+y^2)}{2\ln 2}$$

is also a solution.

1A.5 Green's identities

Consider the divergence theorem:

$$\int_D \nabla \cdot \underset{\sim}{F}\, dV = \oint_{\partial D} \underset{\sim}{F} \cdot \hat{n}\, dS$$

If $\underset{\sim}{F} = \nabla u$ then we get Green's first identity:

$$\int_D \nabla^2 u \, dV = \oint \underset{\sim}{\nabla u} \cdot \underset{\sim}{n} \, dS$$

If $F = v \nabla u - u \nabla v$ then we get Green's second identity (*)

$$\left.\begin{aligned} \int_D \nabla \cdot (v \nabla u - u \nabla v) \, dV &= \int_D (v \nabla^2 u - u \nabla^2 v) \, dV \\ &= \oint_{\partial D} (v \underset{\sim}{\nabla u} \cdot \underset{\sim}{n} - u \nabla v \cdot n) \, dS . \end{aligned}\right\} (*)$$

finally, we can integrate by parts

$$\int_D v \nabla^2 u \, dV = \int_D v \, \nabla \cdot (\nabla u) \, dV$$

$$= \int_D \nabla \cdot (v \nabla u) - \nabla u \cdot \nabla v \, dV$$

$$= \int_{\partial D} v \underset{\sim}{\nabla u} \cdot \underset{\sim}{n} \, dS - \int_D \nabla u \cdot \nabla v \, dV$$

so that

$$\int_D \underset{\sim}{\nabla u} \cdot \underset{\sim}{\nabla v} \, dV = \int_{\partial D} v \, \underset{\sim}{n} \cdot \nabla u \, dS - \int_D v \nabla^2 u \, dV .$$

$\Rightarrow$ the third Green's identity.

6.1.6 Application of Green's identity to the "uniqueness" of Neumann problems

Consider the problem $\nabla^2 u = f(x,y) \quad (x,y) \in D$.

$$\underset{\sim}{n} \cdot \nabla \underset{\sim}{u} = g(x,y) \quad (x,y) \in \partial D$$

then given two solutions v_1 and v_2 to the problem, construct

$$v = v_1 - v_2.$$

Then v solves
$$\begin{cases} \nabla^2 v = 0 & (x,y) \in D \\ \underset{\sim}{n} \cdot \nabla \underset{\sim}{v} = 0 & (x,y) \in \partial D. \end{cases}$$

Use Green's third identity with $u = v$ then

$$\int_V v \nabla^2 v \, dV = \int_{\partial V} v \hat{n} \cdot \nabla \underset{\sim}{v} \, dS - \int_V |\nabla v|^2 \, dV$$

$$\Rightarrow \int_V |\nabla v|^2 \, dV = 0 \rightarrow \nabla v = 0 \text{ everywhere}$$

$$\Rightarrow v \text{ is } \underline{\text{constant}}.$$

So if v_1 is a solution then any other function

$$v = v_1 + K \text{ is also a solution}$$

Exercise: What happens in the case of Robin conditions?

$$\nabla^2 u = f(x,y) \quad (x,y) \in D.$$

$$u + \alpha \hat{n} \cdot \nabla u = g(x,y) \quad (x,y) \in \partial D$$