so the solution to the problem is $y(x) = \int_0^1 G(x,x')F(x')dx'$

$$y(x) = \int_0^1 \sum_n{}' \frac{3\sin(2\pi x')}{1-n^2\pi^2} 2\sin(n\pi x)\sin(n\pi x')\,dx'$$

$$= \frac{3}{1-4\pi^2}\sin(2\pi x)$$

6.7.2 Application to parabolic/hyperbolic PDEs.

Now consider either $u_t - \frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = F(x,t)$

or $u_{tt} - \frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = F(x,t)$.

Idea: Solve the associated Sturm-Liouville problem

$$\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] + \lambda u = 0$$

to find the eigenvalues and eigenfunctions

$\{v_n\}$, $\{\lambda_n\}$

then expand

$$F(x;t) = \sum_n{}' b_n(t) v_n(x)$$

(in this case, $b_n(t) = \int_a^b F(x,t) r(x) v_n(x)\,dx$)

Assume a solution of the form

$$u(x,t) = \sum_n{}' a_n(t) v_n(x)$$

and try the ansatz into the equation:

Parabolic case: $\sum_n{}' \dot{a}_n(t) v_n(x) - \frac{1}{r(x)}\left[\left(p(x)\sum_n a_n(t) v_n'(x)\right)' + q(x)\sum a_n(t) v_n(x)\right]$

$$= \sum{}' b_n(t) v_n(x)$$

so that

$$\sum_n{}' \dot{a}_n(t)\, v_n(x) + \sum_n{}' \lambda_n a_n(t)\, v_n(x) = \sum_n b_n(t)\, v_n(x)$$

and (by orthogonality):

$$\dot{a}_n + \lambda_n a_n = b_n(t)$$

↳ integrating factor method:

$$\frac{d}{dt}\left(a_n e^{\lambda_n t}\right) = b_n(t)\, e^{\lambda_n t}$$

so $$a_n(t)e^{\lambda_n t} - a_n(0) = \int_0^t b_n(t')\, e^{\lambda_n t'}\, dt'$$

$$\Rightarrow a_n(t) = a_n(0)e^{-\lambda_n t} + e^{-\lambda_n t}\int_0^t b_n(t')\, e^{\lambda_n t'}\, dt'.$$

Putting it all together we find that

$$u(x,t) = \sum_n{}' a_n(0)e^{-\lambda_n t} v_n(x) + \int_0^t \sum_n{}' e^{-\lambda_n (t-t')} v_n(x)\, b_n(t')\, dt'$$

$$= \sum_n{}' a_n(0)e^{-\lambda_n t} v_n(x) + \int_0^t\int_a^b \sum_n{}' e^{-\lambda_n (t-t')} v_n(x)\, v_n(x')\, r(x')\, F(x',t')\, dx'\, dt'$$

So we can write

$$u(x,t) = \sum_n a_n(0)e^{-\lambda_n t} v_n(x) + \int_0^t\int_a^b G(x,t;x',t')\, F(x',t')\, dx'\, dt'$$

with $$G(x,t;x',t') = \sum_n{}' e^{-\lambda_n (t-t')} v_n(x)\, v_n(x')\, r(x')$$

Here G is called the Heat Equation Kernel.
↳ another example of a Green's function.

↳ u is the sum of
- the solution to the problem with no forcing

\+ • the weighted integral of $F(x,t)$ with the Green's function.

Example of the drunks exiting the pub.

Recall: $$\frac{\partial p}{\partial t} = k\frac{\partial^2 p}{\partial x^2} + S(x,t)$$

$$\begin{cases} p(x,0)=0 \\ \frac{\partial p}{\partial x} = 0 \text{ at } x=0,L \\ S(x,t) = S_0 e^{-\frac{t}{\tau}} \delta(x-\frac{L}{2}) \quad \text{for } t>0. \end{cases}$$

(take $t_0=0$).

Homogeneous problem; separation of variables to get spatial eigenmodes $\Rightarrow$

$$\begin{cases} V_n(x) = \cos\left(\frac{n\pi x}{L}\right) \\ \lambda_n = \frac{n^2\pi^2}{L^2} \end{cases}$$

So, by the previous calculation, we have

$$u(x,t) = \sum_{n=0}^{\infty} a_n(0)\, e^{-\lambda_n t} V_n(x) + \int_0^t \int_0^L S(x',t')\, G(x,x';t,t')\, dx'dt'$$

where $a_n(0)$ are obtained by fitting u to initial conditions

$$u(x,0) = \sum_{n=0}^{\infty} a_n(0) V_n(x) = 0 \Rightarrow a_n(0) = 0$$

and where $G(x,x';t,t') = \sum_{n=0}^{\infty} e^{-\lambda_n(t-t')} \frac{V_n(x')V_n(x)\, r(x)}{\|V_n^2\|}$ ← but $r(x)=1$

$$\Rightarrow u(x,t) = \int_0^t\int_0^L \sum_{n=0}^{\infty} S_0 e^{-\frac{t'}{\tau}} \delta(x'-\frac{L}{2})\, e^{-\frac{n^2\pi^2}{L^2}(t-t')} \frac{1}{\|V_n^2\|} \cos\left(\frac{n\pi x'}{L}\right)\cos\left(\frac{n\pi x}{L}\right) dx'dt'$$

$$= \int_0^t \sum_{n=0}^{\infty} S_0 e^{-\frac{t'}{\tau}} e^{-\frac{n^2\pi^2}{L^2}(t-t')} \cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{L}\right) dt' \cdot \frac{1}{\|V_n^2\|}$$

$$= \sum_{n=0}^{\infty} S_0 \cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{L}\right) \frac{1}{\frac{n^2\pi^2}{L^2} - \frac{1}{\tau}} \left[e^{-\frac{t}{\tau}} - e^{-\frac{n^2\pi^2}{L^2}t} \right] \frac{1}{\|V_n^2\|}$$

Hyperbolic case : similarly

$$\sum_n' \ddot{a}_n(t)\, v_n(x) + \sum' \lambda_n\, a_n(t)\, v_n(x) = \sum' b_n(t) v_n(x)$$

so by orthogonality

$$\ddot{a}_n(t) + \lambda_n a_n(t) = b_n(t)$$

This time we use the Laplace transform method:

$$s^2 \hat{a}_n - s a_n(0) - a_n'(0) + \lambda_n \hat{a}_n = \hat{b}_n(s)$$

$$\Rightarrow \quad \hat{a}_n(s) = \frac{\hat{b}_n(s) + s a_n(0) + a_n'(0)}{s^2 + \lambda_n}$$

Now, the λ_n are positive $\Rightarrow$ the inverse Laplace transform (see Maple) is

$$a_n(t) = \frac{1}{\sqrt{\lambda_n}} \int_0^t b_n(t') \sin\left(\sqrt{\lambda_n}\,(t-t')\right) dt'$$

$$+ \; a_n(0) \cos\left(\sqrt{\lambda_n}\, t\right) + a_n'(0) \frac{1}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n}\, t\right)$$

$\Rightarrow$ The general solution of the problem becomes

$$u(x,t) = \sum_{n=0}^{\infty} v_n(x) \cdot \left[\frac{1}{\sqrt{\lambda_n}} \int_0^t b_n(t') \sin\left(\sqrt{\lambda_n}\,(t-t')\right) dt' \right.$$

$$\left. + \; a_n(0) \cos\left(\sqrt{\lambda_n}\, t\right) + a_n'(0) \frac{1}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n}\, t\right) \right]$$

but with $b_n(t') = \displaystyle\int_a^b \frac{F(x',t')\, v_n(x')\, r(x')\, dx'}{\|v_n^2\|}$

we get $$u(x,t) = \sum' \left[a_n(0) \cos\left(\sqrt{\lambda_n}\, t\right) + a_n'(0) \frac{1}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n}\, t\right) \right] \cdot v_n(x)$$

$$+ \int_0^t \int_a^b F(x',t')\, G(x,x';t,t')\, dx'\, dt'$$

with

$$G(x,x';t,t') = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n}(t-t')\right) \frac{v_n(x) v_n(x') r(x')}{\| v_n^2 \|}$$

↳ the wave kernel

Example of the bridge

Recall : $u_{tt} - c^2 u_{xx} = \sin\left(\frac{2\pi x}{L}\right)\cos(\omega t)$

$u = 0$ at both ends

$u_t(x,0) = u(x,0) = 0$

$\Rightarrow$ Eigenmodes/values of spatial homogeneous pb:

$$\begin{cases} v_n(x) = \sin\left(\frac{n\pi x}{L}\right) \\ \lambda_n = \frac{n^2\pi^2c^2}{L^2} \end{cases}$$

then $u(x,t) = \sum \left[a_n(0)\cos\left(\frac{n\pi c t}{L}\right) + a_n'(0)\frac{L}{n\pi c}\sin\left(\frac{n\pi c t}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$

$$+ \int_0^t \int_0^L F(x',t')\, G(x,x';t,t')\, dx'\, dt'$$

Fitting this to ICs $\Rightarrow$ $a_n(0) = a_n'(0) = 0$

$$u(x,t) = \int_0^t dt' \int_0^L dx' \sin\left(\frac{2\pi x'}{L}\right)\cos(\omega t') \sum_{n=0}^{\infty} \frac{L}{n\pi c}\sin\left(\frac{n\pi c}{L}(t-t')\right) \frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\frac{L}{2}}$$

$$= \int_0^t dt' \cos(\omega t') \frac{L}{2\pi c}\sin\left(\frac{2\pi c}{L}(t-t')\right)\sin\left(\frac{2\pi x}{L}\right)$$

$$= \int_0^t \frac{dt'}{2}\left[\sin\left(\omega t' + \frac{2\pi c}{L}(t-t')\right) - \sin\left(\omega t' - \frac{2\pi c}{L}(t-t')\right)\right]\sin\frac{2\pi x}{L}\,\frac{L}{2\pi c}$$

$$= \frac{1/2}{\omega + \frac{2\pi c}{L}}\left[\cos\left(\frac{2\pi c t}{L}\right) - \cos\omega t\right] - \frac{1/2}{\omega - \frac{2\pi c}{L}}\left[\cos\left(\frac{2\pi c}{L}\right) - \cos\omega t\right]\frac{L}{2\pi c} \cdot \sin\left(\frac{2\pi x}{L}\right)$$

$$= \frac{1}{\omega^2 - 4\pi^2c^2/L^2}\left[\cos\left(\frac{2\pi c t}{L}\right) - \cos\omega t\right]\sin\left(\frac{2\pi x}{L}\right) \checkmark$$