(8) Consider a regular S.L problem with eigenvalues $\{\lambda_0 \ldots \lambda_n \ldots\}$ and eigenfunctions $\{v_0 \ldots v_n \ldots\}$.

Then v_n has exactly n roots over the interval (a,b)

In particular, v_0 has no node within (a,b)

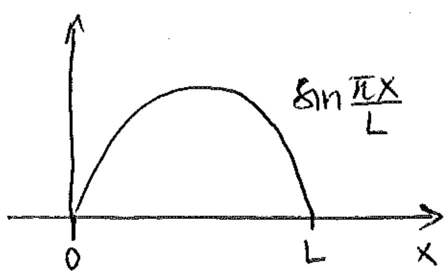
Remark : This is why the simplest guess for $u(x)$ for estimating λ_0 actually is also the best.

Example :

$$\begin{cases} u'' + \lambda u = 0 \\ u(0) = u(L) = 0 \end{cases}$$

$$\begin{cases} \lambda_n = \dfrac{\pi^2}{L^2}(n+1)^2 \\ v_n = \sin\left(\dfrac{\pi x}{L}(n+1)\right) \end{cases} \quad n \geq 0$$

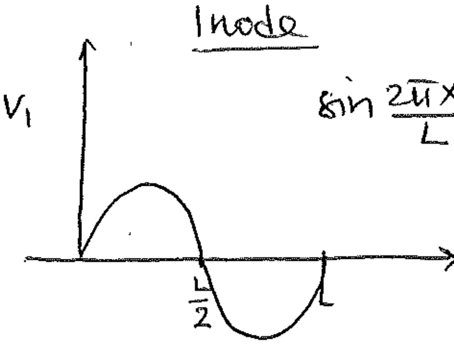
V_0:
0 nodes



$\sin \frac{\pi x}{L}$

1 node

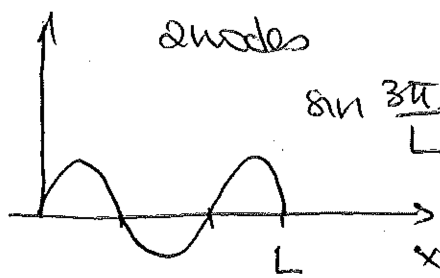
V_1



$\sin \frac{2\pi x}{L}$

2 nodes

V_2



$\sin \frac{3\pi x}{L}$

etc...

(8) Asymptotic ($n \to \infty$) approximations to the eigenfunctions and eigenvalues of a regular SL problem

For large n, we know that $\lambda_n \to +\infty$; if this is the case, it is possible to approximate the eigenfunctions by

$$v_n(x) \simeq \frac{1}{(r(x)p(x))^{1/4}} \left\{ \alpha \cos\left[\sqrt{\lambda_n} \int_a^x \sqrt{\frac{r(x')}{p(x')}}\, dx'\right] + \beta \sin\left[\sqrt{\lambda_n} \int_a^x \sqrt{\frac{r(x')}{p(x')}}\, dx'\right] \right\}$$

(This formula is derived from the WKB approximation (see AMS 212b))

In that case it's easy to see that

$$\sqrt{\lambda_n} \int_a^b \sqrt{\frac{r(x')}{p(x')}}\, dx' \simeq n\pi$$

$$\Rightarrow \qquad \lambda_n \simeq \left(\frac{n\pi}{\int_a^b \sqrt{\frac{r(x')}{p(x')}}\, dx'} \right)^2$$

<u>Example</u>

$$\begin{cases} u'' + \lambda u = 0 \\ u(0) = u'(0) \\ u(1) = -u'(1) \end{cases}$$

we saw that $\lambda \geq 0$

This time, let's look for the eigensolutions:

$$u(x) = A\cos\sqrt{\lambda}\, x + B\sin\sqrt{\lambda}\, x$$

to satisfy the bcs we calculate

$$u'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}\,x + B\sqrt{\lambda}\cos\sqrt{\lambda}\,x$$

so
$$\begin{cases} A = B\sqrt{\lambda} \\ A\cos\sqrt{\lambda} + B\sin\sqrt{\lambda} = +A\sqrt{\lambda}\sin\sqrt{\lambda} - B\sqrt{\lambda}\cos\sqrt{\lambda} \end{cases}$$

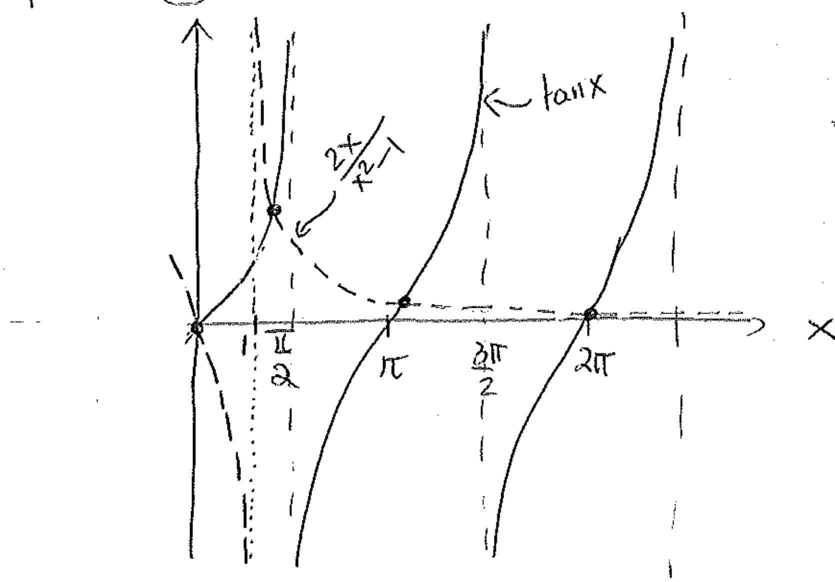
$$\Rightarrow \begin{cases} \sqrt{\lambda}\cos\sqrt{\lambda} + \sin\sqrt{\lambda} = +\lambda\sin\sqrt{\lambda} - \sqrt{\lambda}\cos\sqrt{\lambda}. \\ A = B\sqrt{\lambda} \end{cases}$$

$$\Rightarrow \begin{cases} 2\sqrt{\lambda}\cos\sqrt{\lambda} = (\lambda - 1)\sin\sqrt{\lambda} \\ A = B\sqrt{\lambda} \end{cases}$$

$$\Rightarrow \begin{cases} \tan\sqrt{\lambda} = \dfrac{2\sqrt{\lambda}}{\lambda - 1} \\ A = B\sqrt{\lambda} \end{cases}$$

to find λ, we must solve the equation $\tan x = \dfrac{2x}{x^2 - 1}$

Graphically with $x > 0$



$\Rightarrow$ looks like

$x_n \simeq n\pi$

for large n

$\Rightarrow \lambda_n \simeq n^2\pi^2$

Check : using the asymptotic formula with

$r(x) = p(x) = 1 \quad q(x) = 0 \qquad \begin{matrix} a = 0 \\ b = 1 \end{matrix}$

$\Rightarrow \quad \lambda_n \cong (n^2\pi^2)$. indeed for large n.

6.6 Example of application: wave in a non-homogeneous medium

Consider the wave equation for varying wave speed:

$$\frac{\partial^2 u}{\partial t^2} = c^2(x)\frac{\partial^2 u}{\partial x^2} \qquad c^2(x) > 0 \quad \forall x \in [0,1].$$

on a finite string : $x \in [0,1]$.
with $u(0) = u(1) = 0 \quad \forall t$

Then this is an archetype S.L problem / eigenfunction expansion problem.

let $\quad u = T(t)F(x)$ then

$$\frac{\ddot{T}}{T} = -\lambda \qquad c^2(x)\frac{F''}{F} = -\lambda$$

so $\quad F'' = -\dfrac{\lambda F}{c^2(x)}$, a Sturm-Liouville problem with

$$\begin{cases} p(x) = 1 \\ q(x) = 0 \\ r(x) = \dfrac{1}{c^2(x)} \end{cases}$$

What can we learn from this without actually solving the equations?

① $\underline{\lambda \geq 0}$.

Indeed: $\quad R(u) = \dfrac{\int_0^1 u'^2\,dx}{\int_0^1 \frac{u^2}{c^2(x)}\,dx} \geq 0$ for any function u.

② Some estimate of the fundamental mode of vibration can be made by minimizing $R(u)$: $\quad f_0 = \sqrt{\lambda_0}$ with

$$0 \leq \lambda_0 \leq \frac{\int_0^1 \bar{u}'^2\,dx}{\int_0^1 \frac{\bar{u}^2}{c^2(x)}\,dx}$$

(i.e. try $\bar{u} = \sin(\pi x)$ or $\bar{u} = x(1-x)$)

(3) Some estimate of the high frequencies of vibration can be made: $f_n \simeq \sqrt{\lambda_n}$ with

$$\lambda_n \simeq \left(\frac{n\pi}{\int_0^1 \frac{1}{c(x)}\, dx} \right)^2$$

<u>Example</u>: suppose we consider a string with a slight defect at $x = x_0$

$$c(x) = c_0\left(1 + \epsilon e^{-\frac{(x-x_0)^2}{2\sigma^2}}\right)$$

then
$$\frac{1}{c(x)} \simeq \frac{1}{c_0}\left(1 - \epsilon e^{-\frac{(x-x_0)^2}{2\sigma^2}}\right)$$

and for large n
$$\lambda_n \simeq n^2\pi^2 \left[\int_0^1 \frac{1}{c_0}\left(1 - \epsilon e^{-\frac{(x-x_0)^2}{2\sigma^2}}\right) dx \right]^{-2}$$

- if $\sigma \ll 1$ then the width of the Gaussian is small enough that we can approximate
$$\int_0^1 e^{-\frac{(x-x_0)^2}{2\sigma^2}} dx \simeq \int_{-\infty}^{+\infty} e^{-\frac{(x-x_0)^2}{2\sigma^2}} dx \simeq \sqrt{2\pi}\,\sigma$$

In that case
$$\lambda_n \simeq n^2\pi^2 c_0^2 \left(1 - \sqrt{2\pi}\,\sigma\epsilon\right)^{-2}$$
$$\simeq n^2\pi^2 c_0^2 \left(1 + 2\sqrt{2\pi}\,\sigma\epsilon\right)$$

$\rightarrow$ by comparing the frequencies "observed" on the imperfect string to those from a theoretical "perfect" string, we can deduce $\underline{\sigma\epsilon}$ but not $\overline{x_0}$.

6.7 Non-homogeneous equations; introduction to Green's functions

6.7.1 Non homogeneous (regular) SL problems (ODEs)

Given the ODE $\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = F(x)$

with bcs $\begin{cases} \alpha u(a) + \beta u'(a) = 0 \\ \gamma u(b) + \delta u'(b) = 0 \end{cases}$

① Seek solutions of the homogeneous eigenvalue eq.

$$\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = -\lambda u$$

→ this yields the eigenfunctions $\{v_n\}$ and eigenvalues $\{\lambda_n\}$

② Write $F(x) = \sum_n b_n v_n(x)$

(with $b_n = \int_a^b r(x')F(x')v_n(x')dx'$, if the v_ns are properly normalized)

Then since we know that the solution can also be written as

$$u(x) = \sum_n a_n v_n(x)$$

we can write

$$\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = \sum_n -\lambda_n a_n v_n(x) = \sum_n b_n v_n(x)$$

and by identification, $a_n = -\frac{b_n}{\lambda_n}$

$$\Rightarrow \quad u(x) = \sum_n -\frac{b_n}{\lambda_n} v_n(x) = -\sum_n \int_a^b \frac{r(x')}{\lambda_n} F(x') v_n(x') v_n(x) dx'$$

$$= \int_a^b G(x; x') F(x') dx'$$

where $$G(x;x') = -\sum_n \frac{1}{\lambda_n} v_n(x') v_n(x)\, r(x')$$

- $G(x;x')$ is called the Green's function of the S.L problem
- It only depends on the characteristics of the homogeneous problem ($\{v_n\}$, $\{\lambda_n\}$) but, when integrated through with the forcing term $F(x)$, yields the solution of the forced problem
- Note that if the $\{v_n\}$ are not normalized then

$$G(x;x') = -\sum_n \frac{1}{\|v_n\|^2} \frac{r(x')}{\lambda_n} v_n(x') v_n(x) .$$

where $\|v_n\|^2 = \int_a^b r(x)\, v_n(x)^2\, dx$.

<u>Example</u> Consider

$$y'' + y = 3\sin(2\pi x) \qquad \begin{matrix} y(0) = 0 \\ y(1) = 0 \end{matrix}$$

We seek the eigenfunctions of $y'' + y = -\lambda y$

$$\rightarrow \quad y'' + (1+\lambda) y = 0$$

so $$y = \alpha \cos(\sqrt{1+\lambda}\, x) + \beta \sin(\sqrt{1+\lambda}\, x)$$

with $$\begin{cases} \alpha = 0 \\ \sqrt{1+\lambda_n} = n\pi \end{cases}$$

$$\Rightarrow \begin{cases} \lambda_n = n^2\pi^2 - 1 \\ v_n(x) = \sin(n\pi x) \end{cases}$$

$\Rightarrow$ The Green's function $$G(x,x') = -\sum_n \frac{\sin(n\pi x)\sin(n\pi x')}{\lambda_n \|\sin n\pi x\|^2}$$

$$= -\sum_n \frac{2}{n^2\pi^2 - 1} \sin(n\pi x) \sin(n\pi x')$$