This is an equation for the 0th-order Bessel function

$$A_n(y) = \begin{Bmatrix} J_0(y) \\ Y_0(y) \end{Bmatrix}$$

(See attached handout on Bessel functions)

$$\Rightarrow \quad A_n(x) = \gamma_n J_0(\sqrt{\lambda_n}\, x) + \delta_n Y_0(\sqrt{\lambda_n}\, x)$$

To guarantee regularity at the origin $\delta_n = 0$.

To guarantee $A_n(R) = 0 \Rightarrow J_0(\sqrt{\lambda_n}\, R) = 0$ which implies

$$\sqrt{\lambda_n}\, R = z_n$$

where z_n is the n-th zero of J_0

$$\Rightarrow \quad \lambda_n = \frac{z_n^2}{R^2}$$

So finally,

$$u(x,t) = \sum a_n \sin\left(c\frac{z_n}{R}t\right) J_0\left(\frac{z_n}{R}x\right)$$

where

$$a_n = \frac{R}{c z_n} \frac{\int_0^R x e^{-x^2/2\sigma^2} J_0\left(\frac{z_n}{R}x\right) dx}{\int_0^R x J_0^2\left(\frac{z_n}{R}x\right) dx}$$

(see Maple file)

(7) Define $\quad \mathcal{R}(u) = - \dfrac{\int_a^b u\, \mathcal{L}(u)\, dx}{\int_a^b r u^2\, dx}$

(the Rayleigh Quotient)

then, the following theorem holds,

Theorem: • The principal eigenvalue λ_0 of a regular Sturm-Liouville problem is the solution of

$$\lambda_0 = \inf_{u \in V} R(u) \qquad \text{(Rayleigh-Ritz formula)}$$

where V is the space of all continuous & differentiable functions on (a,b) such that u satisfy the BCs of the Sturm-Liouville problem, and $u \neq 0$ (not the trivial function)

• The function u_0 for which the minimum of $R(u)$ is achieved is the corresponding eigenfunction of the principal eigenvalue

Proof: let $\{\lambda_0 \dots \lambda_n \dots\}$ be the set of all eigenvalues of the S.L problem with $\{v_0 \dots v_n \dots\}$ the set of corresponding orthonormal eigenfunctions

then $$u = \sum_n{}' a_n v_n(x)$$

and $$\mathcal{L}(u) = -\sum_n{}' a_n \lambda_n r(x) v_n(x).$$

Then $$\int_a^b u\mathcal{L}(u)\,dx = \int_a^b -\sum_n{}'\sum_m{}' a_n a_m \lambda_n r(x) v_n(x) v_m(x)\,dx$$

$$= -\sum_n{}'\sum_m{}' \int a_n a_m \lambda_n r(x) v_n(x) v_m(x)\,dx$$

modulo some arguments about exchanging Σ and $\int$

$$= -\sum_n{}' a_n^2 \lambda_n$$

$$\int_a^b r(x) u^2\,dx = \int_a^b \sum_n{}'\sum_m{}' a_n a_m v_n(x) v_m(x) r(x)\,dx$$

same $$= \sum_n{}' a_n^2$$

so $$R(u) = \frac{\sum_n a_n^2 \lambda_n}{\sum_n a_n^2}$$

now given that we know that $\forall n>0$ $\lambda_n > \lambda_0$ then

$$R(u) \geq \frac{\lambda_0 \sum_n a_n^2}{\sum_n a_n^2} = \lambda_0$$

To have equality, we would require that $a_n = 0$ $\forall n>0$ so that

$$R(u) = \frac{a_0^2 \lambda_0}{a_0^2} = \lambda_0$$

If $u_0 = a_0 v_0$ then u_0 is indeed the eigenfunction corresponding to the principle eigenvalue λ_0.

<u>Notes</u> ① given that $\int_a^b u\mathcal{L}(u)\,dx$

$$= \int_a^b u\left[(p(x)u')' + q(x)u \right] dx$$

$$= \int_a^b q(x)u^2\,dx + \left[u p(x) u' \right]_a^b - \int_a^b p(x)u'^2\,dx$$

then $$R(u) = \inf_{u \in V} \left[\frac{\int_a^b (p(x)u'^2 - q(x)u^2)dx - [uu'p]_a^b}{\int_a^b r u^2\,dx} \right]$$

This becomes particularly simple for homogeneous Neumann or Dirichlet conditions: in that case

$$R(u) = \inf_{u \in V} \left[\frac{\int_a^b [p(x)u'^2 - q(x)u^2]dx}{\int_a^b r u^2\,dx} \right]$$

This form is often v. useful to determine the sign of λ_0, and to obtain order-of-magnitude estimates for it

(6.1) <u>Example 1</u>. Consider the S.L problem

$$\begin{cases} u'' + \lambda u = 0 \\ u(0) - u'(0) = 0 \\ u(1) + u'(1) = 0 \end{cases} \qquad x \in [0,1].$$

Here, we have a S.L problem with

$$\begin{cases} p(x) = 1 \\ q(x) = 0 \\ r(x) = 1 \end{cases}$$

so $$R(u) = \frac{\int_0^1 u'^2\,dx - [u(1)u'(1) - u(0)u'(0)]}{\int_0^1 u^2\,dx}$$

but $u'(1) = -u(1)$ and $u'(0) = u(0) \Rightarrow [u(1)u'(1) - u(0)u'(0)] = \begin{matrix}-u(1)^2 \\ -u(0)^2\end{matrix}$

→ so clearly $R(u) \geq 0$ for all u, which proves that $\lambda_0 \geq 0$.

(6.4) <u>Example 2</u> Consider the S.L. problem.

$$\begin{cases} u'' + (\lambda - x^2)u = 0 \\ u'(0) = 0 \\ u(1) = 0 \end{cases}$$

We want to find an estimate for λ_0.

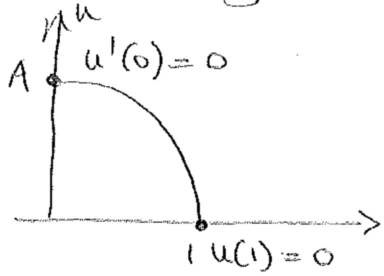
Here, we have the S.L problem with

$$\begin{cases} p(x) = 1 \\ q(x) = -x^2 \\ r(x) = 1. \end{cases} \qquad \text{so}$$

$$R(u) = \frac{\int_0^1 u'^2 + x^2u^2 \, dx - \cancelto{0}{\left[u(1)u'(1) - u(0)u'(0)\right]}}{\int_0^1 u^2 \, dx}$$

so we see that $R(u) \geq 0 \Rightarrow \lambda_0 \geq 0$

To obtain an estimate for λ_0, consider a trial function $u(x)$ that satisfy the boundary conditions



$\rightarrow$ could try $u(x) = A\cos\left(\frac{\pi}{2}x\right)$

or $u(x) = A(1-x^2)$

Using the 2nd option

$$R(u) = \frac{\int_0^1 \cancel{A^2}(-2x)^2 + x^2\cancel{A^2}(1-x^2)^2 \, dx}{\int_0^1 \cancel{A^2}(1-x^2)^2 \, dx}$$

$$= \frac{\int_0^1 (4x^2 + x^2 - 2x^4 + x^6) \, dx}{\int_0^1 1 - 2x^2 + x^4 \, dx}$$

$$= \frac{\frac{5}{3} - \frac{2}{5} + \frac{1}{7}}{1 - \frac{2}{3} + \frac{1}{5}} = \frac{37}{14} \quad \Rightarrow \quad 0 \leq \lambda_0 \leq \frac{37}{14}$$

Exercise; try the same procedure with $A\cos\left(\frac{\pi}{2}x\right)$

<u>Note</u>: True solution has $\lambda_0 = 2.597\ldots$ $\quad \frac{37}{14} = 2.64$

(2) It is actually possible to show that the sequence of eigenvalues of the S.L problem $(pu_n')' + qu_n = -\lambda_n r u_n$ is also the set of <u>all stationary pts</u> of the Rayleigh quotient $R(u)$ over V, and the eigenfunctions are the functions for which this stationary pt is achieved:

$$\lambda_n = R(v_n)$$