5.4 Introduction to Sturm-Liouville Pbs.

- The eigenvalue problem

$$(p(x)u')' + q(x)u + \lambda r(x)u = 0$$

on the open interval $x \in (a,b)$
with

$$\begin{cases} \alpha u(a) + \beta u'(a) = 0 \\ \gamma u(b) + \delta u'(b) = 0 \end{cases}$$

is called a Sturm-Liouville problem provided
- $p(x)$, $p'(x)$, $q(x)$ and $r(x)$ are continuous
- $p(x)$, $r(x)$ > 0 in (a,b)

and
- $|\alpha| + |\beta| > 0$, $|\gamma| + |\delta| > 0$

- if $p(x)$ or $r(x)$ vanish at $x=a$ or $x=b$, or if the interval (a,b) is unbounded (i.e either a or b $\to \pm\infty$) then the problem is called a <u>singular</u> Sturm-Liouville problem; otherwise the problem is <u>regular</u>

- The function $r(x)$ is called the <u>weight function</u>

<u>Examples</u>

① $\begin{cases} \dfrac{d^2u}{dx^2} + \lambda u = 0 \\ u(0) = u(L) = 0 \end{cases}$ is a regular S.L problem with $p(x)=1$ $q(x)=0$ $r(x)=1$

Bessel eq. ② $\begin{cases} x^2 \dfrac{d^2u}{dx^2} + x\dfrac{du}{dx} + (x^2 - \nu^2)u = 0 \quad r \in (0, +\infty) \\ |u(0)| < +\infty \quad u(L) = 0 \end{cases}$ is a <u>singular</u> S.L problem with $r(x) = +\frac{1}{x}$ $p(x) = x$ $q(x) = x$ $\lambda = -\nu^2$

- Note that we may also consider periodic S.L problems: where $u(a) = u(b)$ and $u'(a) = u'(b)$ are the BCs.

5.5 Properties of Sturm-Liouville problems (ODEs)

① Symmetry of the operator

Given two functions u and v satisfying

$$\begin{cases} \alpha v(a) + \beta v'(a) = 0 \\ \gamma v(b) + \beta v'(b) = 0 \end{cases} \qquad \begin{cases} \alpha u(a) + \beta u'(a) = 0 \\ \gamma u(b) + \beta u'(b) = 0 \end{cases}$$

then $$\int_a^b [u\mathcal{L}(v) - v\mathcal{L}(u)]\,dx = 0$$

Proof $$\int_a^b [u\mathcal{L}(v) - v\mathcal{L}(u)]\,dx$$

$$= \int_a^b \{ u[(pv')' + qv] - v[(pu')' + qu] \} dx$$

$$= \int_a^b \{ u(pv')' - v(pu')' \} dx$$

integrate by parts.

$$\left[upv' \right]_a^b - \int_a^b pu'v'\,dx - \left[pvu' \right]_a^b + \int_a^b pu'v'\,dx$$

$$= \left[p(uv' - vu') \right]_a^b$$

$$= p(b)\{ u(b)v'(b) - v(b)u'(b) \} - p(a)\{ u(a)v'(a) - v(a)u'(a) \}$$

$$= 0 \qquad \text{using the bcs.}$$

② Orthogonality of the eigenfunctions

Eigenfunctions corresponding to $\neq$ eigenvalues λ are orthogonal wrt the inner product

$$\langle u, v \rangle = \int_a^b u(x) v(x) r(x)\,dx.$$

Proof: Let u_n be an eigenfunction with λ_n e.value

u_m ——————— with λ_m e.value

$$\Rightarrow \begin{cases} \mathcal{L}(u_n) = -\lambda_n r u_n \\ \mathcal{L}(u_m) = -\lambda_m r u_m \end{cases}$$

then $\displaystyle\int_a^b \left[u_m \mathcal{L}(u_n) - u_n \mathcal{L}(u_m)\right] dx = 0$ by symmetry

$$= \int_a^b (\lambda_m - \lambda_n) r \, u_n u_m \, dx$$

$$= (\lambda_m - \lambda_n) \langle u_n, u_m \rangle$$

so unless $\lambda_m = \lambda_n$, $\langle u_n, u_m \rangle = 0$ $\square$

③ The eigenvalues of the Sturm-Liouville problem are real

Proof: Suppose λ is a complex eigenvalue, corresponding to a complex solution 'u'.

then $\mathcal{L}(u) = -\lambda r u = (pu')' + qu$

then taking the c.c on both sides $\Rightarrow$

$\mathcal{L}(u^*) = -\lambda^* r u^*$ $\Rightarrow \lambda^*$ is the eigenvalue corresponding to the eigenfunction u^*.

$\Rightarrow$ if $\lambda \notin \mathbb{R}$ then $\lambda \neq \lambda^*$ and so

$\langle u, u^* \rangle = 0$

But $\displaystyle\int_a^b u u^* r \, dx = \int_a^b |u|^2 r \, dx > 0$ unless u is identically 0.

$\rightarrow$ So we reach a contradiction, implying that $\lambda \in \mathbb{R}$.

(4) The eigenvalues of a regular Sturm Liouville problem are simple i.e: if two functions have the same eigenvalue then these functions are linearly dependent.

Proof: let v_1 and v_2 be two eigenfunctions belonging to the same eigenvalue

$$\mathcal{L}(v_1) = \lambda r v_1$$
$$\mathcal{L}(v_2) = \lambda r v_2$$

$$\Rightarrow \quad v_2 \mathcal{L}(v_1) = v_1 \mathcal{L}(v_2) = \lambda r v_1 v_2$$

$$\text{so} \quad v_2 \mathcal{L}(v_1) - v_1 \mathcal{L}(v_2) = 0 \quad \text{for all } x.$$

Recall that
$$v_2 \mathcal{L}(v_1) - v_1 \mathcal{L}(v_2) = v_2 [(p v_1')' + q v_1] - v_1 [(p v_2')' + q v_2]$$
$$= v_2 (p v_1')' - v_1 (p v_2')'$$
$$= (p (v_2 v_1' - v_1 v_2'))'$$

$$\text{so} \quad v_2 v_1' - v_1 v_2' = \text{constant}.$$

However, on the boundaries this quantity is 0

$$\Rightarrow \quad v_2 v_1' = v_1 v_2'$$

$$\Rightarrow \left(\frac{v_1}{v_2}\right)' = 0 \Rightarrow \boxed{v_1 = \alpha v_2}$$

(5) The set of all eigenvalues for a regular Sturm-Liouville problem forms an unbounded, strictly monotone sequence:

$$\lambda_0 < \lambda_1 < \lambda_2 \cdots < \lambda_n < \lambda_{n+1} < \cdots < +\infty.$$

and $\lim_{n \to \infty} \lambda_n = +\infty$; λ_0 is called the principal eigenvalue

(6) It is possible to construct a set of eigenfunctions $\{V_n\}$ of a regular Sturm-Liouville problem in such a way that

+ all eigenfunctions in the set are real
+ they are orthonormal w.r.t the inner product

$$\langle V_n, V_m \rangle = \int_a^b V_n(x) V_m(x)\, r(x)\, dx$$

+ the set is a <u>complete basis</u> for all piecewise continuous functions defined on the interval $[a,b]$, so that these functions can be written as the convergent series

$$f(x) = \sum_{n=0}^{\infty} a_n V_n(x) \qquad \forall x \in [a,b].$$

with
$$a_n = \int_a^b f(x) V_n(x) r(x)\, dx$$

(note: if V_n are not normalized, then
$$a_n = \frac{\int_a^b f(x) V_n(x) r(x)\, dx}{\int_a^b V_n^2(x) r(x)\, dx}.$$
)

$\Rightarrow$ <u>Generalized Fourier Series</u>

<u>Examples</u>

<u>Example (A)</u> The Fourier functions.

let
$$\begin{cases} \dfrac{d^2u}{dx^2} = -\lambda u \\ u(0) = u(L) = 0 \end{cases}$$

We know that this is a Sturm Liouville problem (regular) with $p(x)=1$, $r(x)=1$, $q(x)=0$.

The eigenvectors and eigenvalues are

$$\begin{cases} V_n(x) = \sin\left(\frac{n\pi x}{L}\right) \\ \lambda_n = \frac{n^2\pi^2}{L^2} \end{cases} \quad (n = 1 \ldots \infty)$$

let's redefine $m = n-1$ so that $\begin{cases} \lambda_m = \frac{(m+1)^2\pi^2}{L^2} \quad (m = 0 \ldots \infty) \\ V_m(x) = \sin\left(\frac{(m+1)\pi x}{L}\right) \end{cases}$

Let's verify each of the properties again:

(a) Symmetry of the operator

$$\int_0^L \left(\frac{d^2u}{dx^2} v - \frac{d^2v}{dx^2} u\right) dx \quad \text{for } u, v \text{ satisfying the same bcs}$$

$$= \left[v \frac{du}{dx}\right] - \int_0^L \frac{du}{dx}\frac{dv}{dx} dx$$

$$- \left[u \frac{dv}{dx}\right] + \int_0^L \frac{du}{dx}\frac{dv}{dx} dx$$

$$= \left[v\frac{du}{dx} - u\frac{dv}{dx}\right]_0^L = 0$$

(b) Orthogonality of the eigenfunctions

assume $n \neq m$:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \int_0^L \left[\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)\right]\frac{dx}{2}$$

$$= \frac{L}{(n-m)\pi} \sin\left(\frac{(n-m)\pi x}{L}\right)\Bigg]_0^L - \frac{L}{(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{L}\right)\Bigg]_0^L$$

$$= 0$$

if $n = m$ then

$$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = \int_0^L \left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right)\frac{dx}{2}$$

$$= \frac{L}{2}$$

(c) λ is indeed real

(e) $\{\lambda_n\}$ form an infinite monotonous sequence

(f) Every function f satisfying $f(0) = f(L) = 0$ can be written as

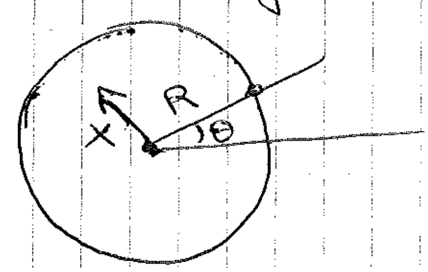
$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

with $$a_n = \frac{\langle f(x), \sin\left(\frac{n\pi x}{L}\right)\rangle}{\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\rangle}$$

$$= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$ $\rightarrow$ see previous chapter.

Example (B) Bessel functions.

Let's consider a circular drum, assume it is oscillating in an axisymmetric way:



$$u_{tt} = \frac{c^2}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)$$

$$u(R,t) = 0$$
$$|u(0,t)| < +\infty$$

At time $t=0$, we hit the drum dead center with a stick, giving it a velocity $u_t(x,0) = e^{-x^2/2}$

$$\begin{cases} u(x,0) = 0 \\ u_t(x,0) = e^{-x^2/2} \end{cases}$$

Separation of variables : $u(x,t) = A(x)B(t)$

$$\begin{cases} \dfrac{d^2B}{dt^2} = -c^2\lambda B \\ \dfrac{1}{x}\dfrac{d}{dx}\left(x\dfrac{dA}{dx}\right) = -\lambda A \end{cases}$$

The second equation represents a singular Sturm–Liouville problem with

$$\begin{cases} p(x) = x \\ q(x) = 0 \\ r(x) = x \end{cases}$$

Let the solutions be $A_n(x)$ with eigenvalues $\{\lambda_n\}$. Then, the associated $B_n(t)$ is

$$B_n(t) = \alpha_n \cos\left(c\sqrt{\lambda_n}\, t\right) + \beta_n \sin\left(c\sqrt{\lambda_n}\, t\right)$$

Given that $u(x,0) = 0 \Rightarrow \alpha_n = 0$

So

$$u(x,t) = \sum_{n=0}^{\infty} a_n \sin\left(c\sqrt{\lambda_n}\, t\right) A_n(x).$$

and to find the a_n, use the last IC:

$$u_t(x,0) = e^{-x^2/2}$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n c\sqrt{\lambda_n}\, A_n(x) = e^{-x^2/2}$$

Since the $\{A_n(x)\}$ form an orthogonal basis on $[0,R]$ with the weight function $r(x) = x$, we know that

$$a_n c\sqrt{\lambda_n} = \frac{\langle e^{-x^2/2}, A_n\rangle}{\langle A_n, A_n\rangle}$$

with $\displaystyle \langle u,v\rangle = \int_0^R x\, u(x) v(x)\, dx$.

What are the $\{A_n(x)\}$ and the $\{\lambda_n\}$?

Note that

$$\frac{d}{dx}\left(x\frac{dA}{dx}\right) = -\lambda A x \iff y^2\frac{d^2A}{dy^2} + y\frac{dA}{dy} + y^2 A = 0$$

with $y = \sqrt{\lambda}\, x$