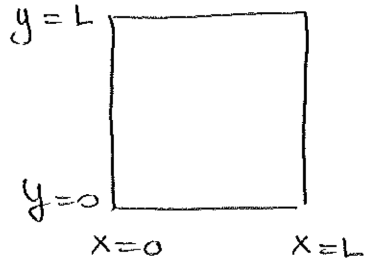
(3) Poisson equation

Suppose we want to solve $\nabla^2 T = -H(x,y)$

↑ heating source

to obtain the steady-state temperature profile in a metallic plate, heated as prescribed by $H(x,y)$ and with $T=0$ on all 4 sides; take $k=1$



Note that the − sign comes from

$$\frac{\partial T}{\partial t} = \nabla^2 T + H(x,y)$$

→ in steady state $\nabla^2 T = -H$.

The spatial eigenmodes in x-direction are (see previous lectures), for $T(0,y) = T(L,y) = 0$

$$A_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

→ Assume $T(x,y) = \sum_{n=1}^{\infty} A_n(x) B_n(y)$

then

$$\sum_{n=1}^{\infty} -\frac{n^2\pi^2}{L^2} A_n(x) B_n(y) + A_n(x) \frac{d^2 B_n}{dy^2} = -H(x,y)$$

Noting that $\int_0^L A_n(x) A_m(x)\, dx = \frac{L}{2} \delta_{mn}$,

$$\frac{d^2 B_n}{dy^2} - \frac{n^2\pi^2}{L^2} B_n = -\frac{2}{L} \int_0^L H(x,y) \sin\left(\frac{n\pi x}{L}\right) dx.$$

$$= -h_n(y).$$

Suppose that $H(x,y) = \delta\left(x-\frac{L}{2}\right)\delta\left(y-\frac{L}{2}\right)$ to model a point source

Then $$\frac{d^2 B_n}{dy^2} - \frac{n^2\pi^2}{L^2} B_n = -\sin\left(\frac{n\pi}{2}\right) \delta\left(y - \frac{L}{2}\right) \cdot \frac{2}{L}$$

Laplace transforms, review

Laplace transforms are very useful for solving non-homogeneous linear ODES.

Idea: let f be a function of t

The Laplace transform of f is

$$\mathcal{L}(f) = \hat{f}(p) = \int_0^\infty f(t) e^{-pt}\, dt.$$

Properties: $\mathcal{L}(f') = p\hat{f}(p) - f(0)$

since $$\int_0^\infty \frac{df}{dt} e^{-pt}\, dt = \left[f e^{-pt} \right]_0^\infty + \int_0^\infty p f e^{-pt}\, dt$$
$$= p\hat{f}(p) - f(0)$$

$$\mathcal{L}(f'') = p^2 \hat{f}(p) - p f(0) - f'(0)$$
(proof is similar).

So given a linear ODE with constant coefficients

$$af'' + bf' + cf = g(t) \qquad (*)$$

$$\mathcal{L}(*) \Rightarrow \quad a\left[p^2 \hat{f}(p) - pf(0) - f'(0)\right]$$
$$+ \quad b\left[p\hat{f}(p) - f(0)\right]$$
$$+ \quad c\hat{f}(p) = \int_0^\infty g(t) e^{-pt}\, dt. = G(p)$$

Suppose $f(0)$ and $f'(0)$ are known (initial value problem) then this is an algebraic equation for $\hat{f}(p)$.

To recover $f(t)$, we need to do an inverse Laplace transform.

For detail on Inverse Laplace transforms, see handout. Usually, it's easy to find the solution using Inverse Laplace transform tables.

Here:

$$\frac{d^2 B_n}{dy^2} - \frac{n^2\pi^2}{L^2} B_n = -\sin\left(\frac{n\pi}{2}\right) \delta\left(y - \frac{L}{2}\right) \cdot \frac{2}{L}$$

$$\Rightarrow \quad p^2 \hat{B}_n - p B_n(0) - B_n'(0) - \frac{n^2\pi^2}{L^2} \hat{B}_n = -\frac{2}{L} \sin\left(\frac{n\pi}{2}\right) \int_0^\infty \delta\left(y - \frac{L}{2}\right) e^{-py}\, dy$$

$$= -\sin\left(\frac{n\pi}{2}\right) e^{-p\frac{L}{2}} \cdot \frac{2}{L}$$

$B_n(0) = 0$ but $B_n'(0)$ is unknown. Let's leave it as is for the moment.

$$\Rightarrow \quad \hat{B}_n \left[p^2 - \frac{n^2\pi^2}{L^2} \right] = B_n'(0) - \frac{2}{L} \sin\left(\frac{n\pi}{2}\right) e^{-p\frac{L}{2}}$$

so

$$\hat{B}_n(p) = \frac{B_n'(0)}{p^2 - \frac{n^2\pi^2}{L^2}} - \frac{2}{L} \sin\left(\frac{n\pi}{2}\right) e^{-p\frac{L}{2}} \frac{1}{p^2 - \frac{n^2\pi^2}{L^2}}$$

From tables:

- The inverse transform of $\frac{1}{p^2 - a^2}$ is $\frac{\sinh(ay)}{a}$
- The inverse transform of

$$\frac{e^{-pb}}{p^2 - a^2} \quad \text{is} \quad \begin{cases} \frac{\sinh(a(y-b))}{a} & \text{if } y > b \\ 0 & \text{if } 0 < y < b \end{cases}$$

$$\Rightarrow \quad B_n(y) = \frac{B_n'(0)}{\frac{n\pi}{L}} \cdot \sinh\left[\frac{n\pi y}{L}\right]$$

$$- \frac{2}{L}\sin\left(\frac{n\pi}{2}\right) \sinh\left(\frac{n\pi(y-\frac{L}{2})}{L}\right)\frac{1}{\frac{n\pi}{L}} \quad \text{if} \quad y > \frac{L}{2}$$

At $y = L$, the solution is such that $B_n(L) = 0 \Rightarrow$

$$B_n'(0)\sinh(n\pi) - \frac{2}{L}\sin\left(\frac{n\pi}{2}\right)\sinh\left(\frac{n\pi}{2}\right) = 0$$

$$\Rightarrow \quad B_n'(0) = \frac{2}{L}\frac{\sin\left(\frac{n\pi}{2}\right)\sinh\left(\frac{n\pi}{2}\right)}{\sinh(n\pi)}.$$

So finally, we have

$$T(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \frac{\sin\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{L}} \frac{2}{L}\left[\frac{\sinh\left(\frac{n\pi}{2}\right)}{\sinh(n\pi)}\sinh\left(\frac{n\pi y}{L}\right) - \sinh\left(\frac{n\pi(y-\frac{L}{2})}{L}\right) \underbrace{H\left(y-\frac{L}{2}\right)}_{\text{Heaviside function}}\right]$$

<u>Note</u>: This expression is slightly awkward, but it can be shown that it indeed leads to the correct behavior in y, which should be symmetry across the $y = \frac{L}{2}$ line. (Homework!)

CHAPTER 5. Generalization of separation of variables: Sturm-Liouville theory & eigenfunction expansions

In Chapter 4, we studied some very simple linear PDEs (with constant coefficients) with simple boundary conditions (rectangular domains) which lend themselves particularly well to the separation of variables

In this chapter, we generalize the method to any homogeneous linear PDE of the form

$$\begin{cases} m(t)\, u_t = \mathcal{L}_x^{(2)}(u) \\ m(t)\, u_{tt} = \mathcal{L}_x^{(2)}(u) \end{cases} \quad \text{where } \mathcal{L}_x^{(2)}(u) = a(x)u_{xx} + b(x)u_x + c(x)u$$

and formalize the notion of boundary conditions.

5.1 Separation of variables in this case

Let, as usual, $u(x,t) = A(x)B(t)$ then we have (in the parabolic case, for example)

$$\frac{m(t)}{B}\frac{dB}{dt} = \frac{1}{A}\left[a(x)\frac{d^2A}{dx^2} + b(x)\frac{dA}{dx} + c(x)A\right] = \text{constant}.$$

$$\rightarrow \begin{cases} \dfrac{dB}{dt} = \dfrac{KB}{m(t)} \\ a(x)\dfrac{d^2A}{dx^2} + b(x)\dfrac{dA}{dx} + c(x)A = KA \end{cases}$$

For a given set of boundary conditions,
The x-equation is an eigenvalue problem, which typically has an infinite number of solutions $A_n(x)$ each associated with a particular value K_n

$A_n(x)$ is called an eigen-mode
K_n is the associated eigen-value

The eigenmodes characterize the intrinsic spatial properties of the PDE. The eigenvalues characterize its intrinsic temporal properties.
(see previous chapter for examples).

5.2 Classification of the boundary conditions

Since we may be interested in a variety of domain shapes and associated BCs, we need a new classification system.

For a given spatial domain Ω, we can apply the following BCs to the contour $\partial\Omega$ of the domain:

(a) <u>Dirichlet conditions</u>

$$u(\underline{r},t) = f(\underline{r},t) \quad \forall\ r \in \partial\Omega$$

i.e the value of the function is fixed on the contour

<u>examples</u> • $u(\underline{r},t) = 0$ (null condition)

cf. Guitar string pinned at $x=0$ and $x=L$ (edge of domain)

• $u(\underline{r},t) = K$ (constant condition)

cf. Ends of a rod held at some temperature K

(b) <u>Von Neumann conditions</u>

for $\underline{r} \in \partial\Omega$, $\underline{n} \cdot \nabla u = f(\underline{r},t)$ where $\underline{n}$ is the vector normal to the contour/edge of the domain.

i.e. the <u>flux</u> of u through the boundary is fixed.

<u>example</u>: $\dfrac{\partial u}{\partial z} = 0$ at $z = 0, L$.

(c) <u>Robin conditions = mixed conditions</u>

$$\alpha(\underline{r},t)\, \underline{n} \cdot \nabla u + \beta(\underline{r},t)\, u(r,t) = f(r,t) \quad \forall\ \underline{r} \in \partial\Omega.$$

Note: • This nomenclature applies to domains in any number of dimensions. (spatial.

➾ For a 1D interval, then

Dirichlet conditions on [ab]: $\begin{cases} u(a,t) = u_1(t) \\ u(b,t) = u_2(t) \end{cases}$

Neumann conditions on [a,b]: $\begin{cases} \frac{\partial u}{\partial x}(a,t) = u_1(t) \\ \frac{\partial u}{\partial x}(b,t) = u_2(t) \end{cases}$

Robin conditions: $\begin{cases} \alpha u(a,t) + \beta u_x(a,t) = u_1(t) \\ \gamma u(b,t) + \delta u_x(b,t) = u_2(t) \end{cases}$

$|\alpha| + |\beta| > 0 \quad |\gamma| + |\delta| > 0.$

6.3 Reformulation of the PDE

We now reformulate the problem by

Saying $\mathcal{L}_x^{(2)} = a(x)u_{xx} + b(x)u_x + c(x)u$ with $a \neq 0$

Multiply by $\frac{p(x)}{a(x)}$ with $p(x) = e^{\int \frac{b(x)}{a(x)} dx}$

then

$$\frac{p(x)}{a(x)} \mathcal{L}_x^{(2)} = p(x)u_{xx} + \frac{p(x)}{a(x)} b(x) u_x + \frac{c(x)}{a(x)} p(x) u$$

$$= p(x) u_{xx} + \frac{dp}{dx} u_x + \frac{c(x)}{a(x)} p(x) u$$

$$= (p u_x)_x + \frac{c(x)}{a(x)} p(x) u.$$

since $\frac{dp}{dx} = \frac{b(x)}{a(x)} e^{\int \frac{b(x)}{a(x)} dx} = \frac{b(x)}{a(x)} p(x)$

So the original PDES can be rewritten as

$$u_t = \frac{1}{m(t)r(x)}\left[(p(x)u_x)_x + q(x)u \right]$$

where $\quad r(x) = \dfrac{p(x)}{a(x)}$

$$q(x) = \frac{c(x)}{a(x)}\,p(x)$$

$$\left(\text{and } \; p(x) = e^{\int \frac{b(x)}{a(x)}dx} \right)$$

and similarly for the u_{tt} case.

As a result, separation of variables leads to (for the parabolic case, for example)

$$\begin{cases} \dfrac{dB}{dt} = \dfrac{KB(t)}{m(t)} \\[2ex] \dfrac{1}{r(x)}\left[\dfrac{d}{dx}\left(p(x)\dfrac{dA}{dx}\right) + q(x)A\right] = KA \end{cases}$$

let $K = -\lambda$ (a simple re-definition) then

$$\begin{cases} \dfrac{dB}{dt} = -\lambda\dfrac{B(t)}{m(t)} \\[2ex] \dfrac{d}{dx}\left[p(x)\dfrac{dA}{dx}\right] + q(x)A = -\lambda r(x)A \end{cases}$$

(similarly for the hyperbolic case)

The x-equation is a special type of eigenvalue ODE called a Sturm-Liouville equation, which has been extensively studied mathematically and for which there exist many important results.