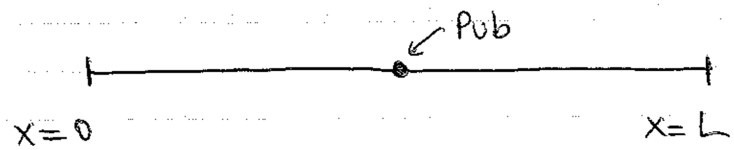
② Forced diffusion equation

A pub in England rings last orders at 11:00 pm, at which point people start to leave and go back home. They are only "locals", i.e, people living in the same 1D street. Being quite drunk, they walk randomly in the street although they dont leave it. We assume they can't find their keys and stay in the street a long time ---



← Street of length L.

Pub at $x=\frac{L}{2}$

⊕ We model the evolution of the population density in the street as a diffusion process:

$$\Rightarrow \quad \frac{\partial p}{\partial t} = k\frac{\partial^2 p}{\partial x^2} + S(x,t)$$

- population density $p = p(x,t)$.
- $S(x,t) =$ # of people / unit time being released in the street by the pub.

⊕ At $t = t_0$ the street is empty

⊕ To model the "they don't leave the street" idea, we use insulating boundary conditions.

$$\Rightarrow \quad \frac{\partial p}{\partial x} = 0 \quad \text{at both boundaries.}$$

⊕ To model the flux of people out of the pub, we assume

$$S(x,t) = S_0 e^{-(t-t_0)/\tau} \, \delta\left(x - \frac{L}{2}\right)$$

$t_0 =$ last orders time

$\tau \simeq$ time till closing, say 1/2 hour.

$\delta =$ a delta function

Recall: $\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a)$.

Solution 1. Find the spatial eigenmodes of the homogeneous problem.

- From previous lectures, we know that

$$\begin{cases} A_0(x) = a_0 x + b_0 \\ A_n(x) = a_n \cos(\lambda_n x) + b_n \sin(\lambda_n x) \end{cases} \quad (\lambda \text{ to be determined})$$

to satisfy $\frac{dp}{dx} = 0$ at both ends we need

- $n \neq 0$ $\quad \frac{dA_n}{dx} = \lambda_n\left(-a_n \sin(\lambda_n x) + b_n \cos(\lambda_n x)\right)$

$$\Rightarrow \begin{cases} \left.\frac{dA_n}{dx}\right|_{x=0} = 0 \Rightarrow b_n = 0 \\ \left.\frac{dA_n}{dx}\right|_{x=L} = 0 \Rightarrow \lambda_n = \frac{n\pi}{L} \end{cases} \Rightarrow A_n(x) = \cos\left(\frac{n\pi x}{L}\right) \text{ (ignore constant)}$$

- $n = 0$ $\quad A_0(x) = \text{constant} = 1$ (ignore constant).

3. Note that $\int_0^L A_n(x) A_m(x)\, dx = \frac{L}{2}\delta_{mn} + \frac{L}{2}\delta_{m0}\delta_{mn}$

2. Suppose the solution is

$$p(x,t) = \sum_0^\infty A_n(x) B_n(t) \text{ and plug into PDE}$$

$$\Rightarrow \sum_0^\infty A_n(x)\dot{B}_n(t) = k\sum_0^\infty -\frac{n^2\pi^2}{L^2} A_n(x) B_n(t) + S(x,t)$$

multiply by $A_m(x)$, integrate on $[0,L]$...

- $m \neq 0$ $\quad \frac{L}{2}\dot{B}_m(t) = -\frac{m^2\pi^2 k}{L^2}\cdot\frac{L}{2} B_m(t) + \int_0^L S(x,t) A_m(x)\, dx$

Now $\int_0^L A_m(x) S_0 e^{-(t-t_0)/\tau}\,\delta(x - \frac{L}{2})\, dx$

$$= S_0 e^{-(t-t_0)/\tau} A_m\left(\tfrac{L}{2}\right) = S_0 e^{-(t-t_0)/\tau}\cos\left(\frac{m\pi}{2}\right)$$

- $m = 0$: $\quad L\dot{B}_0(t) = + S_0 e^{-\frac{t-t_0}{\tau}} \Rightarrow B_0(t) = b_0 - \frac{\tau}{L} S_0 e^{-\frac{t-t_0}{\tau}}$

$\Rightarrow$ The set of decoupled ODEs for the B_n are

$$\dot{B}_n + \frac{n^2\pi^2 k}{L^2} B_n = \frac{2}{L} S_0 e^{-(t-t_0)/\tau} \cos\left(\frac{n\pi}{2}\right)$$

$\rightarrow$ the general solution of the homogeneous problem is

$$B_n^G(t) = \alpha_n e^{-\frac{n^2\pi^2 k}{L^2} t}$$

The particular solution: try $B_n^{PS}(t) = K e^{-\frac{t-t_0}{\tau}}$

$$\Rightarrow \quad -\frac{1}{\tau} K + \frac{n^2\pi^2 k}{L^2} K = \frac{2}{L} S_0 \cos\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \quad K = \frac{\frac{2}{L} S_0 \cos\left(\frac{n\pi}{2}\right)}{\frac{n^2\pi^2 k}{L^2} - \frac{1}{\tau}}$$

So finally, we have $B_n(t) = \alpha_n e^{-\frac{n^2\pi^2 k}{L^2} t} + \dfrac{\frac{2}{L} S_0 \cos\left(\frac{n\pi}{2}\right)}{\frac{n^2\pi^2 k}{L^2} - \frac{1}{\tau}} e^{-\frac{(t-t_0)}{\tau}}$ for $(n \neq 0)$

$\Rightarrow$ $p(x,t) = \sum_{n=0}^{\infty} A_n(x) B_n(t)$ is the complete solution, where the α_ns remain to be determined.

At $t = t_0$ $p(x,t) = 0$ (the street is empty before $t = t_0$)

$$\Rightarrow \quad \sum_{n=0}^{\infty} A_n(x) B_n(0) = 0$$

$$b_0 - \frac{\tau}{L} S_0 e^{-\frac{t-t_0}{\tau}} + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L}\right)\left[\alpha_n e^{-\frac{n^2\pi^2 k t_0}{L^2}} + \frac{\frac{2}{L} S_0 \cos\left(\frac{n\pi}{2}\right)}{\frac{n^2\pi^2 k}{L^2} - \frac{1}{\tau}}\right] = 0$$

$$\Rightarrow \quad \alpha_n = \frac{-S_0 \cos\left(\frac{n\pi}{2}\right) \cdot \frac{2}{L}}{\frac{n^2\pi^2 k}{L^2} - \frac{1}{\tau}} e^{\frac{n^2\pi^2 k t_0}{L^2}} \quad \text{and} \quad b_0 = \frac{\tau}{L} S_0$$

$$\Rightarrow p(x,t) = \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L}\right) \frac{\frac{2}{L} S_0 \cos\left(\frac{n\pi}{2}\right)}{\frac{n^2\pi^2 k}{L^2} - \frac{1}{\tau}} \left(-e^{-\frac{n^2\pi^2}{L} k (t-t_0)} + e^{-(t-t_0)/\tau}\right)$$

$$+ \frac{\tau}{L} S_0 \left(1 - e^{-\frac{t-t_0}{\tau}}\right)$$

Note ① The total number of people in the street at any time is easily derived from the PDE

$$\Rightarrow \quad \frac{\partial p}{\partial t} = k\frac{\partial^2 p}{\partial x^2} + S(x,t)$$

$$\hookrightarrow \frac{\partial}{\partial t}\int_0^L p(x,t)\,dx = k\int_0^L \frac{\partial^2 p}{\partial x^2}\,dx + \int_0^L S(x,t)\,dx$$

$$= k\left[\frac{\partial p}{\partial x}\right]_0^L + S_0 e^{-\frac{t-t_0}{\tau}}$$

$$= S_0 e^{-\frac{t-t_0}{\tau}}$$

So

$$\int_0^L p(x,t)\,dx = \int_{t_0}^{t} S_0 e^{-\frac{t'-t_0}{\tau}}\,dt'$$

$$= \tau S_0\left[1 - e^{-\frac{t-t_0}{\tau}}\right] \quad (t > t_0)$$

↳ at any time the # of people in the street is equal to the total # which has already left the pub. as expected.

② See movies:

– if $\tau \ll \frac{L^2}{\pi^2 k}$ then a large # of people are rapidly released, and then diffuse away from pub entrance

– if $\tau \gg \frac{L^2}{\pi^2 k}$ then the diffusion is faster than release & the people are always ~ evenly spread in the street.

③ Note the "resonnance" between $\frac{1}{\tau}$ and $\frac{n^2\pi^2 k}{L^2}$

$\Rightarrow$ if $\tau \ll \frac{L^2}{\pi k}$ then $\exists\, n$ such that $\frac{1}{\tau} \simeq \frac{n^2\pi^2 k}{L^2}$

That n determines the typical initial "width" of the people density function. (see movie) as $\Delta = \frac{L}{n\pi}$