

4.4 Linear, forced equations

We now consider forced linear PDEs of the kind

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = F(x,t) \quad (\text{Forced Wave Equation})$$

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = F(x,t) \quad (\text{Forced Heat Equation})$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x,y) \quad (\text{"Forced Laplace equation"} \\ = \text{Poisson equation})$$

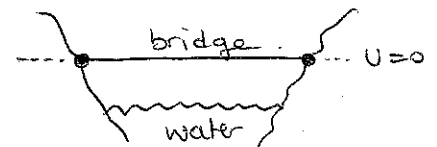
The method for solving these problems will be illustrated through examples.

① Forced wave equation

Example: A bridge, suspended, and the wind forcing (cf Tacoma Narrows)

(in 2D: a metal plate, with some sand on it, and a speakerphone nearby; see Explorationium).

$$\text{let } \begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = F(x,t) \\ u(x, t=0) = 0 & u(0, t) = 0 \\ u_t(x, t=0) = 0 & u(L, t) = 0 \end{cases}$$



→ A forced string, pinned at the sides, initially at rest.

Method: 1. Find the spatial eigenmodes $A_n(x)$ homogeneous problem - with same BCs

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

These will generally be mixtures of sines and cosines.

Here (see previous lectures)

$$A_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

2. Assume that the full solution can be written as

$$u(x,t) = \sum_{n=0}^{\infty} A_n(x) B_n(t)$$

and plug into PDE

$$\Rightarrow \sum_{n=0}^{\infty} A_n(x) \frac{d^2 B_n}{dt^2} - c^2 B_n(t) \frac{d^2 A_n}{dx^2} = F(x,t)$$

$$\Rightarrow \sum_{n=0}^{\infty} A_n(x) \frac{d^2 B_n}{dt^2} + \frac{c^2 n^2 \pi^2}{L^2} B_n(t) A_n(x) = F(x,t) \quad (*)$$

3. Note that $\int_0^L A_n(x) A_m(x) dx = 0 \quad \forall n \neq m$
 $= \frac{L}{2}$ if $n=m$

Aside: The orthogonality property of the eigenfunctions will be true for a wide class of problems, see next chapter.

So take (*) and multiply by $A_m(x)$, then integrate over $[0, L]$

$$\Rightarrow \frac{L}{2} \frac{d^2 B_m}{dt^2} + \frac{c^2 m^2 \pi^2}{L^2} \frac{L}{2} B_m = \int_0^L F(x,t) \sin\left(\frac{m\pi x}{L}\right) dx.$$

$$\Rightarrow \ddot{B}_m + \frac{c^2 m^2 \pi^2}{L^2} B_m = \frac{2}{L} \int_0^L F(x,t) \sin\left(\frac{m\pi x}{L}\right) dx \\ = f_m(t).$$

\Rightarrow we now get a set of independent ODEs, one for each value of m . These are forced, second-order linear ODEs. (which you should be able to solve...)

Simple example

$$\text{Suppose } F(x,t) = \sin\left(\frac{2\pi x}{L}\right) \cos(\omega t)$$

$$\begin{aligned} \text{then } f_m(t) &= \int_0^L \frac{2}{L} \sin\left(\frac{2\pi x}{L}\right) \cos(\omega t) \sin\left(\frac{m\pi x}{L}\right) dx \\ &= \int_0^L \frac{2}{L} \cdot \frac{L}{2} \cdot \cos\pi t \quad \text{if } m=2 \\ &\quad \left. \begin{array}{l} \\ \\ \end{array} \right\} 0 \text{ otherwise} \end{aligned}$$

⇒ we have 2 types of ODEs to solve:

$$\ddot{B}_2 + \frac{4c^2\pi^2}{L^2} B_2 = \cos\omega t$$

and $\ddot{B}_m + \frac{c^2 m^2 \pi^2}{L^2} B_m = 0 \quad \forall m \neq 2.$

For all of them, the solution to the homogeneous equation is

$$B_m(t) = \alpha_m \cos\left(\frac{cm\pi}{L}t\right) + \beta_m \sin\left(\frac{m\pi c}{L}t\right)$$

For the $B_2(t)$ function, we have to add a particular solution to the forced problem: here try

$$B_2^{PS}(t) = k \cos\omega t$$

↑
a constant to be determined by plugging into equation

$$\Rightarrow -k\omega^2 + \frac{4c^2\pi^2}{L^2} k = 1$$

$$\Rightarrow k = \frac{1}{\frac{4c^2\pi^2}{L^2} - \omega^2}$$

$$\Rightarrow B_2(t) = \alpha_2 \cos\left(\frac{2c\pi t}{L}\right) + \beta_2 \sin\left(\frac{2c\pi t}{L}\right) + \frac{\cos(\omega t)}{\frac{4c^2\pi^2}{L^2} - \omega^2}$$

⇒ So finally, we have

$$u(x,t) = \sum_{n=1}^{\infty} A_n(x) B_n(t) \quad \text{where } A_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$$B_n(t) = \alpha_n \cos\left(\frac{n\pi ct}{L}\right) + \beta_n \sin\left(\frac{n\pi ct}{L}\right) + \frac{c \cos \pi t}{4c^2 \pi^2 - \omega^2} \delta_{n,2}$$

To find the arbitrary constants α_n, β_n , we fit the initial conditions:

$$u(x,0) = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} A_n(x) B_n(0) = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} A_n(x) \left[\alpha_n + \frac{1}{4c^2 \pi^2 - \omega^2} \delta_{n,2} \right] = 0$$

$$\Rightarrow \alpha_n + \frac{1}{4c^2 \pi^2 - \omega^2} \delta_{n,2} = 0$$

$$\Rightarrow \begin{cases} \alpha_n = 0 & \text{if } n \neq 2 \\ \alpha_2 = -\frac{1}{4c^2 \pi^2 - \omega^2} \end{cases}$$

$$u_t(x,0) = 0 \Rightarrow \sum_{n=1}^{\infty} A_n(x) \dot{B}_n(0) = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} A_n(x) \left[\beta_n \frac{n\pi c}{L} \right] = 0$$

$$\Rightarrow \beta_n = 0 \quad \forall n.$$

So: the solution becomes very simple!

$$u(x,t) = \left[-\frac{1}{\frac{4c^2\pi^2}{L^2} - \omega^2} \cdot \cos\left(\frac{2\pi ct}{L}\right) + \frac{\cos\omega t}{\frac{4c^2\pi^2}{L^2} - \omega^2} \right] \sin\left(\frac{2\pi x}{L}\right)$$

We note: • $F(x,t) = \cos\omega t \sin\left(\frac{2\pi x}{L}\right)$

specifically forces the system in one of its spatial eigenmodes

→ then this eigenmode is the only one to be excited (see solution).

If $F(x,t)$ had a more complex spatial structure, other modes would be excited too.

• The solution responds by oscillating at two different frequencies simultaneously:

Beating phenomenon. $\left\{ \begin{array}{l} - \text{at the } \underline{\text{forcing}} \text{ frequency} \\ - \text{at the } \underline{\text{intrinsic}} \text{ frequency of the eigenmode.} \end{array} \right.$

• The amplitude of the response goes like

$$\frac{1}{\frac{4c^2\pi^2}{L^2} - \omega^2}$$

intrinsic frequency squared forcing frequency square

→ if the forcing frequency approaches the intrinsic frequency then the amplitude of the response can be huge

This phenomenon is called resonance.

Note: That doesn't mean the amplitude can ever be ∞ : instead if $\omega = \frac{4c^2\pi^2}{L^2}$ the amplitude of the mode grows L^2 linearly with time. (Homework: prove this).