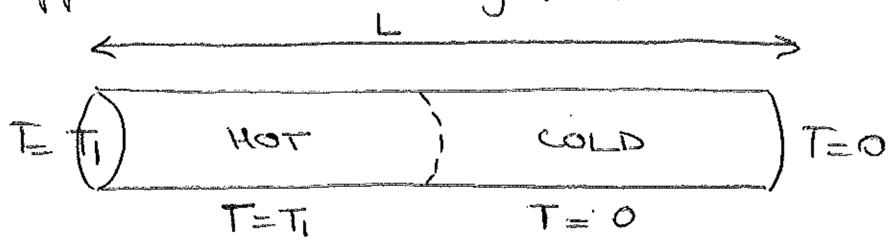
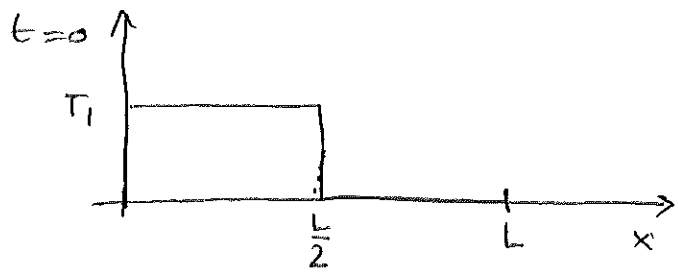
4.2 Heat diffusion in a rod.

Suppose we initially have a rod half-heated



The side walls are insulated so that heat can only be transferred laterally (x-direction).



The edges are kept at temperatures 0 and T_1 respectively.

$$\begin{cases} T(0, t) = T_1 \\ T(L, t) = 0 \end{cases}$$

The PDE is $\quad \dfrac{\partial T}{\partial t} = D \dfrac{\partial^2 T}{\partial x^2}$

Again, we try separating the variables such that

$$T(x, t) = A(x) B(t)$$

$$\Rightarrow \quad A(x) \frac{dB}{dt} = D B(t) \frac{d^2 A}{dx^2}$$

$$\Rightarrow \quad \frac{1}{B} \frac{dB}{dt} = \frac{D}{A} \frac{d^2 A}{dx^2} = \text{constant } K.$$

So
$$\begin{cases} \dfrac{dB}{dt} = KB \\[2ex] \dfrac{d^2 A}{dx^2} = \dfrac{KA}{D} \end{cases}$$

$\longrightarrow$ as before we expect K to be negative to satisfy the boundary conditions simultaneously, so that $K = -k^2$.

Separation of variables

$$T(x,y) = A(x)\,B(y)$$

$$\Rightarrow \quad \frac{1}{A}\frac{d^2A}{dx^2} = K$$

$$\frac{1}{B}\frac{d^2B}{dy^2} = -K$$

Note that

- if $K>0$ then $\begin{cases} A \text{ has exponential behaviour} \\ B \text{ has oscillatory behaviour} \end{cases}$
- if $K=0$ then both must be linear
- if $K<0$ then $\begin{cases} A \text{ has oscillatory behaviour} \\ B \text{ has exponential behaviour} \end{cases}$

- looking at the boundary conditions in x ($A(0)=A(L)=0$) we see that if A is a linear combination of $e^{\sqrt{K}x}$ and $e^{-\sqrt{K}x}$ then the only solution is $A=0$

$$\rightarrow \quad K \leq 0$$

- We can rule out $K=0$ on the same ground

$\rightarrow$ $K<0$ so define $K=-k^2$

$\Rightarrow$ for each k,

$$A_k(x) = a_k \cos kx + b_k \sin kx$$

$$B_k(y) = \alpha_k e^{ky} + \beta_k e^{-ky}$$

or equivalently

$$= \tilde{\alpha}_k \cosh(ky) + \tilde{\beta}_k \sinh(ky)$$

$A_k(0) = A_k(L) = 0 \quad \Rightarrow \quad a_k = 0 \qquad k = \frac{n\pi}{L}$

$B_k(0) = 0 \quad \Rightarrow \quad \tilde{\alpha}_k = 0$

$\rightarrow$ for each value of k there is a possible solution

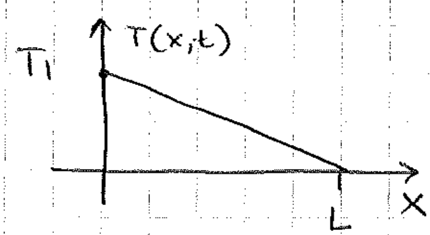
$$A_k(x) = \alpha_k \cos\left(\frac{k}{\sqrt{D}}x\right) + \beta_k \sin\left(\frac{k}{\sqrt{D}}x\right), \quad B_k(t) = e^{-k^2 t}$$

Important Note: if $K = 0$ then there is also a solution with

$$A = ax+b, \quad B = \text{constant}$$

To fit the boundary conditions, let us use our intuition about the problem

- we expect that as $t \to \infty$ the system relaxes to a temperature profile



$T(x, t\to\infty) = T_1 - \frac{T_1}{L}x$

$\rightarrow$ that's the $ax+b$ part!

- The behaviour of $\frac{dB}{dt} = -k^2 B$ suggests decaying exponential modes for all $k \neq 0$

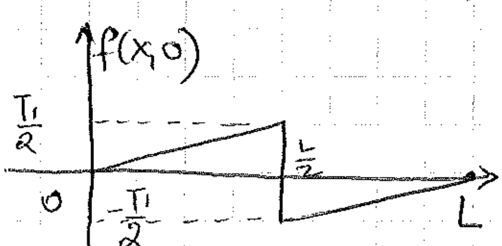
$\Rightarrow$ We expect the solution to be

$$T(x,t) = T_1 - \frac{T_1}{L}x + \begin{pmatrix}\text{some spatial} \\ \sin/\cos \text{ mode}\end{pmatrix} \cdot \begin{pmatrix}\text{a decaying} \\ \text{exponential}\end{pmatrix}$$

$$= T_1 - \frac{T_1}{L}x + f(x,t)$$

where $\begin{cases} f(0,t) = 0 \\ f(L,t) = 0 \end{cases}$

and $f(x,0) = T(x,0) - \left[T_1 - \frac{T_1}{L}x\right]$.



$\rightarrow$ Now we see that if $f(0,t)=0$ then $\alpha_k = 0$ and if $f(L,t) = 0$ then

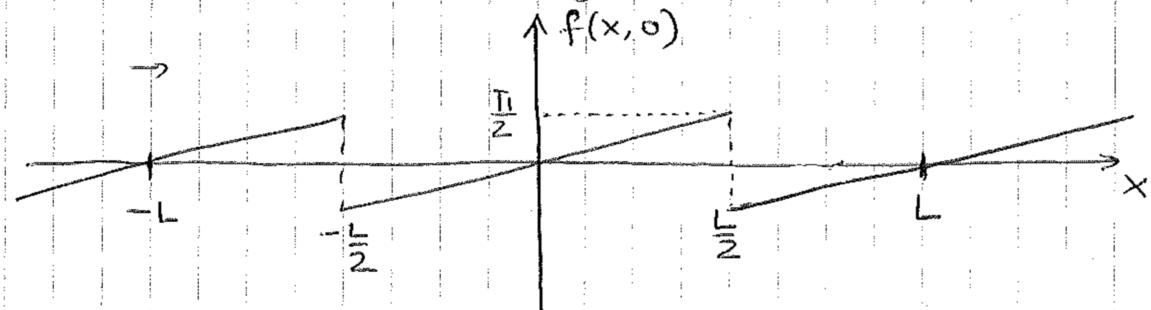
$$\frac{k}{\sqrt{D}}L = n\pi \quad \Rightarrow \quad k = \frac{n\pi\sqrt{D}}{L}$$

So the solution for $T(x,t)$ is

$$T(x,t) = T_1 - \frac{T_1}{L}x + \sum' b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2 Dt}{L^2}}$$

where $\displaystyle\sum_{n=1}^{\infty}{}' b_n \sin\left(\frac{n\pi x}{L}\right) = f(x,0)$

$\Rightarrow$ Construct a Fourier Series for $f(x,0)$ by assuming
- it is periodic with period $2L$
- it is antisymmetric



So $$b_n = \frac{1}{L}\int_{-L}^{L} f(x,0)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^L f(x,0)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^{\frac{L}{2}} \frac{T_1 x}{L}\sin\left(\frac{n\pi x}{L}\right)dx$$

$$+ \frac{2}{L}\int_{\frac{L}{2}}^{L} \frac{T_1}{L}(x-L)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2T_1}{L^2}\left[x\left(\frac{-L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\right]_0^L + \frac{2T_1}{L^2}\left(+\frac{L}{n\pi}\right)\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx$$

$$- \frac{2T_1}{L}\left[\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\right]_{\frac{L}{2}}^{L}$$

$$= -\cancel{\frac{2T_1}{n\pi}\cos(n\pi)} + \cancel{\frac{2T_1}{n\pi}\cos n\pi} - \frac{2T_1}{n\pi}\cos\left(\frac{n\pi}{2}\right)$$

so $b_n = -\dfrac{2T_1}{n\pi}\cos\left(\dfrac{n\pi}{2}\right)$

$$\Rightarrow T(x,t) = T_1\left[1 - \frac{x}{L} + \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\right)\cos\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\pi^2 Dt}{L^2}}\right]$$

Note:

Each mode decays with a different typical timescale: the decay time for mode n is

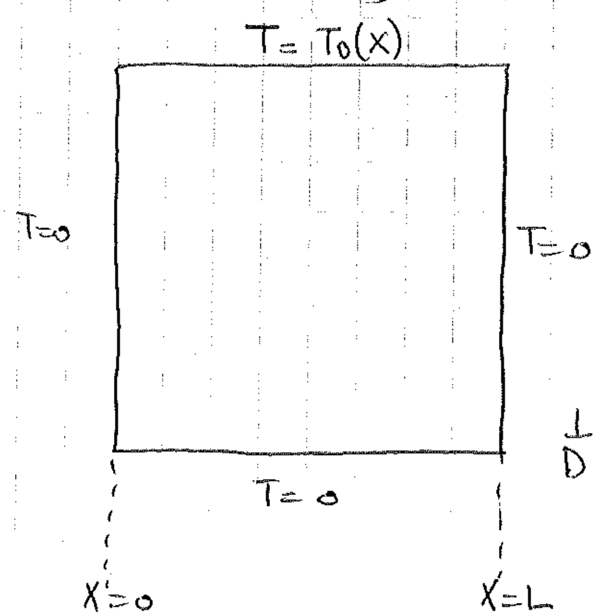
$$\tau_n = \frac{L^2}{\pi^2 n^2 D}$$

→ the higher the degree n, the faster the decay

→ diffusion smoothes out small scale faster than large scales.

4.3 Laplace Equation

Consider a square plate with sides held at the following temperatures:



What is the steady-state temperature profile on the plate as a result of this heating?

→ Solve

$$\frac{1}{D}\frac{\partial T}{\partial t} = \boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0}$$

So $$T(x,y) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)$$

(note that the term for $n=0$ is 0).

To satisfy the remaining boundary condition at $y=1$ we require that

$$T(x,1) = T_0(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi}{L}\right)$$

$\Rightarrow$ This looks like a Fourier Series for an odd function periodic with period $2L$ $\Rightarrow$ let's construct the $\tilde{T}_0(x)$ extension of $T_0(x)$ with these properties, then

$$\sinh\left(\frac{n\pi}{L}\right) b_n = \frac{1}{L}\int_{-L}^{L} \tilde{T}_0(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{\sinh\left(\frac{n\pi}{L}\right)} \frac{2}{L} \int_0^L T_0(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Example: Let $T_0(x) = A\sin^2\left(\frac{\pi x}{L}\right)$ then

$$\frac{2}{L}\int_0^L A\sin^2\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{\cancel{2}}{L}\int_0^L A\left(\frac{1-\cos\left(\frac{2\pi x}{L}\right)}{\cancel{2}}\right)\sin\left(\frac{n\pi x}{L}\right) dx$$

Using $\cos a \sin b = \frac{1}{2}\sin(a+b) - \frac{1}{2}\sin(a-b)$

$$= \frac{A}{L}\left\{ \left[-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L - \frac{1}{2}\int_0^L \sin\left(\frac{(n+2)\pi x}{L}\right) dx + \frac{1}{2}\int_0^L \sin\left(\frac{(2-n)\pi x}{L}\right) dx \right\}$$

$$= \frac{A}{L}\left\{ \left(-\frac{L}{n\pi}\cos(n\pi) + \frac{L}{n\pi}\right) - \frac{1}{2}\frac{L}{(n+2)\pi}\left((-1)^{n+2} - 1\right) - \frac{1}{2}\frac{L}{(n-2)\pi}\left((-1)^{n-2} - 1\right) \right\}$$

if $n \neq 2$.

$\Rightarrow$ if n is even, $b_n = 0$.

if n is odd then

$$b_n = \frac{A}{L}\left\{ + \frac{2L}{n\pi} + \frac{L}{(n+2)\pi} + \frac{L}{(n-2)\pi} \right\} \cdot \frac{1}{\sinh\left(\frac{n\pi}{L}\right)}$$

$$= \frac{1}{\sinh\left(\frac{n\pi}{L}\right)}\left[\frac{2}{n\pi} + \frac{2n}{(4-n^2)\pi}\right]A = \frac{8A}{n(4-n^2)\pi}\,\frac{1}{\sinh\left(\frac{n\pi}{L}\right)}$$

so finally,

$$T(x,y) = \sum_{\substack{n=1\\ n\,\text{odd}}}^{\infty} \frac{8A}{n(4-n^2)\pi}\sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)\frac{1}{\sinh\left(\frac{n\pi}{L}\right)}$$

$$= \sum_{p=0}^{\infty} \frac{8A \cdot \frac{1}{\sinh\left(\frac{(2p+1)\pi}{L}\right)}}{(2p+1)(4-(2p+1)^2)}\sin\left(\frac{(2p+1)\pi x}{L}\right)\sinh\left(\frac{(2p+1)\pi y}{L}\right)$$

<u>Note</u>: we can see that if $A=0$ ($T_0(x)=0$) then the solution in the domain is identically 0

$\Rightarrow$ This is a property of Laplace's equation: if the bcs are identically 0 on the contour then the solution is 0 everywhere.