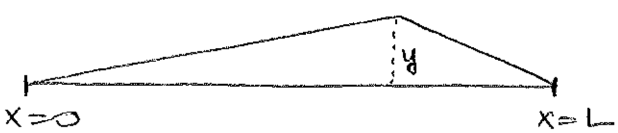
CHAPTER 4 Generic behaviour of Hyperbolic/Parabolic/Elliptic equations through examples. Method of separation of variables

4.1 The wave equation: the vibrating string

Let a string be tightly stretched with ends attached at $x=0$ and $x=L$.



Displacements away from rest are measured by y.

The equation of motion of the string follows a wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

with boundary condition $y(0,t)=0$
$y(L,t)=0$.

Method of separation of variables

Let's assume that we can find a solution with the form

$$y(x,t) = X(x)\,T(t)$$

Then $$X(x)\frac{d^2T}{dt^2} = c^2\,T(t)\frac{d^2X}{dx^2}$$

$$\Leftrightarrow \quad \underbrace{\frac{1}{T(t)}\frac{d^2T}{dt^2}}_{\text{a function of time only}} = \underbrace{\frac{c^2}{X(x)}\frac{d^2X}{dx^2}}_{\text{a function of } x \text{ only}}.$$

Thus equality can only hold for all t, all x provided both sides are constant:

$$\frac{1}{T(t)}\frac{d^2T}{dt^2} = K = \frac{c^2}{X(x)}\frac{d^2X}{dx^2}.$$

This constant is arbitrary for the moment, but it is an eigenvalue of the spatial operator $c^2\frac{d^2}{dx^2}$

$$c^2\frac{d^2X}{dx^2} = K\,X(x)$$

$\Rightarrow$ Will be determined by the boundary conditions.

We want to solve $\begin{cases} \dfrac{d^2X}{dx^2} = \dfrac{K}{c^2}X(x) \\ X(0) = X(L) = 0 \end{cases}$

if $K>0$ then the solutions are exponential
$K<0$ ———————— oscillatory.

if $K>0$ then it is not possible to fit both bcs unless $X(x)=0$ $\forall x$

so let's choose $K<0$, and write $\frac{K}{c^2} = -k^2$

$$\Rightarrow \quad X(x) = A\cos kx + B\sin kx$$

$$X(0)=0 \Rightarrow A=0$$

$$X(L)=0 \Rightarrow \sin kL = 0 \Rightarrow k = \frac{n\pi}{L}$$

So there exist an ∞ of solutions for $X(x)$, namely

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right) \text{ with } n\in\mathbb{Z}$$

Now we must solve $\frac{d^2T}{dt^2} = KT(t) = -c^2k^2T(t)$

so $$T(t) = a\cos(kct) + b\sin(kct)$$

since there are an ∞ of possible values of k then for each $X_n(x)$

correspondes a $T_n(t) = a_n \cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi c t}{L}\right)$

$\Rightarrow$ The general solution to the wave equation with these boundary conditions is a linear combination of all the solutions:

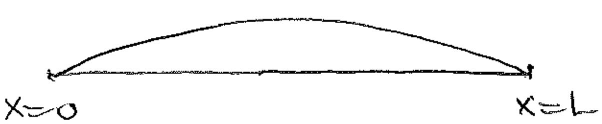
$$y(x,t) = \sum_n X_n(x)\,T_n(t) = \sum_n \sin\left(\frac{n\pi x}{L}\right)\left[\alpha_n \cos\left(\frac{n\pi c t}{L}\right) + \beta_n \sin\left(\frac{n\pi c t}{L}\right)\right]$$

(incorporate a_n with A_n and b_n with B_n + any linear combination coefficient).

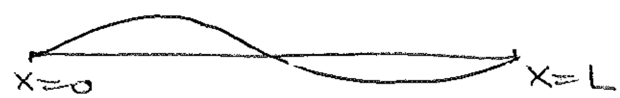
Note : • This expression shows that each spatial function $\sin\left(\frac{n\pi x}{L}\right)$ vibrates with its own frequency $\frac{nc}{2L}$ intrinsic to the system.

$\Rightarrow$ higher c, <u>higher</u> frenquency

$\Rightarrow$ longer L, <u>lower</u> frequency.



$n=1$
(0 nodes)



$n=2$
(1 node)

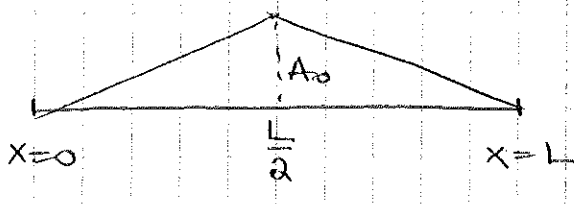
(4 nodes) $n=5$

$\Rightarrow$ Each <u>mode</u> has a different frequency. The higher the <u>degree</u> n of the mode, the higher the frequency (the <u>pitch</u> of the sound emitted for example).

• To determine <u>which</u> mode is excited and with which <u>amplitude</u> it is vibrating, we need to apply initial conditions to the system.

Example: Suppose we pluck the string in the middle, so that at $t=0$ we release it from rest with

$$y(x,0) = \begin{cases} \frac{2A_0 x}{L} & \text{if } 0<x<\frac{L}{2} \\ 2A_0 - \frac{2x}{L}A_0 & \text{if } x \in [\frac{L}{2}, L] \end{cases}$$



$$\frac{\partial y}{\partial t}(x,0) = 0 \quad \text{since we release it from rest}$$

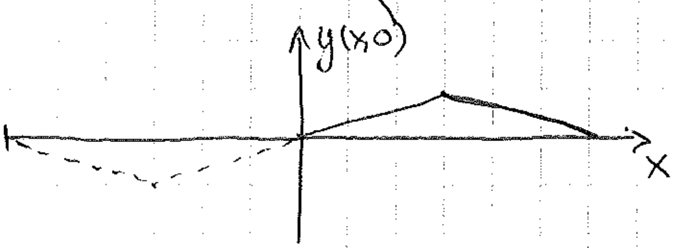
$\Rightarrow$ then we fit the general form $y(x,t)$ to these bcs by requiring that

$$y(x,0) = \sum_n{}' \alpha_n \sin\frac{n\pi x}{L} = \begin{cases} \frac{2A_0 x}{L} & \text{if } x \in [0, \frac{L}{2}] \\ 2A_0 - \frac{2A_0 x}{L} & \text{if } x \in [\frac{L}{2}, L] \end{cases}$$

$$\frac{\partial y}{\partial t}(x,0) = \sum_n{}' \frac{n\pi c}{L}\beta_n \sin\left(\frac{n\pi x}{L}\right) = 0 \Rightarrow \boxed{\beta_n = 0}$$

The first of these two equations implies that we are seeking the coefficients α_n such that the series on the left is equal to the function on the RHS $\Rightarrow$ this looks like a Fourier series problem!

Problem: $y(x,0)$ is not periodic $\Rightarrow$ to remedy the problem, turn $y(x,0)$ into a periodic function by adding the interval $[-L, 0]$



$\rightarrow$ construct an odd function since we are looking for a sin expansion

$\Rightarrow$ By definition $\alpha_n = \frac{1}{L}\int_{-L}^{L} y(x,0)\sin\left(\frac{n\pi x}{L}\right)dx$

by symmetry $= \frac{2}{L}\int_0^L y(x,0)\sin\left(\frac{n\pi x}{L}\right)dx$

$$d_n = \frac{2}{L}\left\{ \int_0^{\frac{L}{2}} \left(\frac{2A_0 x}{L}\right)\sin\frac{n\pi x}{L}\,dx + \int_{\frac{L}{2}}^{L}\left(2A_0 - \frac{2A_0 x}{L}\right)\sin\frac{n\pi x}{L}\,dx\right\}$$

$$= \frac{2}{L}\left\{\left[\frac{2A_0 x}{L}\left(-\frac{L}{n\pi}\right)\cos\frac{n\pi x}{L}\right]_0^{\frac{L}{2}} + \int_0^{\frac{L}{2}}\frac{2A_0}{L}\frac{L}{n\pi}\cos\frac{n\pi x}{L}\,dx\right.$$

$$+ 2A_0\left(-\frac{L}{n\pi}\right)\left[\cos\frac{n\pi x}{L}\right]_{\frac{L}{2}}^{L} - \left[\frac{2A_0 x}{L}\left(-\frac{L}{n\pi}\right)\cos\frac{n\pi x}{L}\right]_{\frac{L}{2}}^{L}$$

$$\left. - \int_{\frac{L}{2}}^{L}\frac{2A_0}{L}\frac{L}{n\pi}\cos\frac{n\pi x}{L}\,dx\right\}$$

$$= \frac{2}{L}\left\{\left[-\frac{2A_0 x}{n\pi}\cos\frac{n\pi x}{L} + \frac{2A_0}{n\pi}\cdot\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^{\frac{L}{2}}\right.$$

$$\left. + \left[-\frac{2A_0 L}{n\pi}\cos\frac{n\pi x}{L} + \frac{2A_0 x}{n\pi}\cos\frac{n\pi x}{L} - \frac{2A_0}{n\pi}\cdot\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right]_{\frac{L}{2}}^{L}\right\}$$

$$= \frac{2}{L}\left\{\cancel{-\frac{2A_0}{n\pi}\frac{L}{2}\cos\left(\frac{n\pi}{2}\right)} + \frac{2A_0 L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right.$$

$$\cancel{-\frac{2A_0 L}{n\pi}\cos(n\pi)} + \cancel{\frac{2A_0 L}{n\pi}\cos\left(\frac{n\pi}{2}\right)} + \cancel{\frac{2A_0 L}{n\pi}\cos(n\pi)}$$

$$\left.\cancel{-\frac{2A_0}{n\pi}\frac{L}{2}\cos\left(\frac{n\pi}{2}\right)} - \cancel{\frac{2A_0 L}{n^2\pi^2}\sin(n\pi)} + \frac{2A_0 L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{2}{L}\left\{\frac{4A_0 L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right\} = \frac{8A_0}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \qquad |n|>1$$

$\Rightarrow$ Finally,

$$y(x,t) = \sum_{n=1}^{\infty}{}' \frac{8A_0}{n^2\pi^2}\sin\left(n\frac{\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

amplitude of the mode — spatial mode — temporal variation of the mode.

The solution is a superposition of all the modes vibrating independently with constant amplitude determined by their initial conditions.

Aside:

a probabilistic derivation of the diffusion equation (Brownian motion)

Imagine a lattice (in 1D). Define the concentration $C(x,t)$ as the expected number of particles at position x, time t.

Particles have equal probability to move left or right (p) and probability to stay where they are ($1-2p$)

So $$C(x,t+\Delta t) = p\left(C(x-\Delta x,t) + C(x+\Delta x,t)\right) + (1-2p)C(x,t)$$

Now assume Δt small and Δx small then

$$C(x,t) + \Delta t \frac{\partial c}{\partial t} = p\left[2C(x,t) + \Delta x^2 \frac{\partial^2 c}{\partial x^2}\right] + (1-2p)C(x,t)$$

$$= C(x,t) + p\Delta x^2 \frac{\partial^2 c}{\partial x^2}$$

$$\Rightarrow \quad \frac{\partial c}{\partial t} = \underbrace{p\frac{\Delta x^2}{\Delta t}}\frac{\partial^2 c}{\partial x^2}$$

define this as the diffusion coefficient k.

Note 1: In the presence of forces this derivation leads to Fokker-Planck equation.

Note 2: From more general considerations, you can derive all the PDEs of fluid mechanics from statistical averaging of ensemble properties of individual particles. Kinetic theory (Boltzmann's equation and its moments).