PARTIAL DIFFERENTIAL EQUATIONS

CHAPTER I: Introduction & Reviews

I Partial differentiation (RHB Ch 5)

① Definitions and examples

- Given a function $f : \mathbb{R}^n \to \mathbb{R}$

$$(x_1, \dots, x_n) \to f(x_1, \dots, x_n)$$

then the partial derivative of f with respect to x_i is

$$\frac{\partial f}{\partial x_i} = \lim_{\epsilon \to 0} \frac{f(x_1 \dots, x_i + \epsilon, \dots x_n) - f(x_1, \dots, x_i, \dots x_n)}{\epsilon}$$

- Notation & properties
  - typically we note $\frac{\partial f}{\partial x_i} = f_{x_i}$
  - we can similarly define higher-order derivatives

$$\text{eg.} \quad f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

  - The order of the partial derivatives is irrelevant:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

- <u>Example</u>: Let $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} \qquad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} \qquad \frac{\partial f}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{-xy}{(x^2+y^2+z^2)^{3/2}} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

(2) Chain rule and changes of variables

- If f is a function of the n variables $(x_1, \dots, x_n)$ and each x_i is a function of the m variables $(u_1 \dots u_m)$ then

$$\frac{\partial f}{\partial u_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial u_j} \quad \text{for } j = 1 \dots m$$

- This property can be used to change variables from one coordinate system to the next

- Example : Cartesian $\leftrightarrow$ Polar

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}(y/x) \end{aligned}$$

$$\left.\frac{\partial f}{\partial r}\right|_\theta = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial r}\right|_\theta + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial r}\right|_\theta$$

notation means at constant θ

$$= \cos\theta \left.\frac{\partial f}{\partial x}\right|_y + \sin\theta \left.\frac{\partial f}{\partial y}\right|_x$$

$$\left.\frac{\partial f}{\partial \theta}\right|_r = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial \theta}\right|_r + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial \theta}\right|_r$$

$$= -r\sin\theta \left.\frac{\partial f}{\partial x}\right|_y + r\cos\theta \left.\frac{\partial f}{\partial y}\right|_x$$

or, going the other way

$$\left.\frac{\partial f}{\partial x}\right|_y = \left.\frac{\partial f}{\partial r}\right|_\theta \left.\frac{\partial r}{\partial x}\right|_y + \left.\frac{\partial f}{\partial \theta}\right|_r \left.\frac{\partial \theta}{\partial x}\right|_y = \frac{x}{r}\left.\frac{\partial f}{\partial r}\right|_\theta - \frac{y}{x^2}\frac{1}{\frac{y^2}{x^2}+1}\left.\frac{\partial f}{\partial \theta}\right|_r$$

$$= \cos\theta \left.\frac{\partial f}{\partial r}\right|_\theta - \frac{\sin\theta}{r}\left.\frac{\partial f}{\partial \theta}\right|_r$$

III Review: First order (first degree) ODES (RHB Ch 14)

Before attacking the problem of solving PDEs, let's review ODES.

3.1 Standard forms

let y be a function of the independent variable x only: $y(x)$

$y =$ dependent variable

First order ODES for $y(x)$ can be written in the equivalent forms

$$\frac{dy}{dx} = F(x, y)$$

or $$A(x,y)\,dx + B(x,y)\,dy = 0$$

$$\left(\text{with } F(x,y) = -\frac{A}{B}\right)$$

3.2 Types of equation & method of solution

3.2.1 Separable variables

Suppose $F(x,y) = f(x)\,g(y)$

or $A(x,y) = a(x)\,\alpha(y)$
$B(x,y) = b(x)\,\beta(y)$

Then

$$\frac{dy}{dx} = F(x,y) \Rightarrow \frac{dy}{g(y)} = dx \cdot f(x)$$

which can be integrated respectively in y and x:

$$\int \frac{dy}{g(y)} = \int dx\, f(x).$$

→ Need to know your integrals; see handout

Example $\quad \dfrac{dy}{dx} = \dfrac{4y}{x(y-3)}$

$$\Rightarrow \quad \frac{y-3}{4y}\,dy = \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{y-3}{4y}\,dy = \int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{4}y - \frac{3}{4}\ln y + K = \ln x$$

K: arbitrary constant of integration to be fitted to boundary condition

Note: Although a solution was found, it is not always possible to write it out easily as $y(x)$.

3.2.2 Exact equations

Suppose we try to solve the form

$$A(x,y)\,dx + B(x,y)\,dy = 0$$

In some cases, this may be the exact differential of a function $U(x,y)$:

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$$

if it so happens that $\begin{cases} A(x,y) = \dfrac{\partial U}{\partial x} \\ B(x,y) = \dfrac{\partial U}{\partial y} \end{cases}$ then $dU = 0$

and the solution is simply $U = \text{constant}$.

How do we know this is an exact differential?

If $A = \dfrac{\partial U}{\partial x}$ and $B = \dfrac{\partial U}{\partial y}$ then $\boxed{\dfrac{\partial A}{\partial y} = \dfrac{\partial B}{\partial x}}$

$\Rightarrow$ Method :

Given the equation $A(x,y)dx + B(x,y)dy = 0$

① Test if exact : calculate $\frac{\partial A}{\partial y}$ and $\frac{\partial B}{\partial x}$

② If exact then find the function $U(x,y)$ such that

$$A(x,y) = \frac{\partial U}{\partial x}$$
$$B(x,y) = \frac{\partial U}{\partial y}$$

③ Set this function to an arbitrary constant to obtain solution

Example : $y\,dx + x\,dy = 0$ $\quad A(x,y) = y$, $B(x,y) = x$

① $\frac{\partial A}{\partial y} = 1 = \frac{\partial B}{\partial x} \Rightarrow$ this is an exact differential

② What is $U(x,y)$ satisfying

$$\begin{cases} \frac{\partial U}{\partial x} = y & : \text{try } U = xy + \text{a function of } y \\ \frac{\partial U}{\partial y} = x & : \text{try } U = xy + \text{a function of } x \end{cases}$$

$\Rightarrow \quad U = xy + \text{constant}$

③ Set $U = \text{const}$ to get solution so

$U(x,y) = K$

$\Rightarrow \quad xy = K'$ (another constant)

$\Rightarrow \quad y = \frac{K'}{x}$

3.2.3 Linear equations

- Linear first order ODEs can always be written as

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

- Suppose we could find a function $\mu(x)$ such that

$$\mu(x)\frac{dy}{dx} + \mu(x)\,P(x)\,y = \frac{d}{dx}\big(\mu(x)\,y\big) \qquad (*)$$

Then the linear ODE would become

$$\frac{d}{dx}\big(\mu(x)\,y\big) = \mu(x)\,Q(x)$$

which can be formally integrated in x to give the solution

$$\mu(x)\,y(x) - \mu(0)\,y(0) = \int_0^x \mu(x')\,Q(x')\,dx'$$

- To find $\mu(x)$, we need to find a function satisfying $(*)$:

$$\cancel{\mu(x)\frac{dy}{dx}} + \mu(x)\,P(x)\,y = \cancel{\mu(x)\frac{dy}{dx}} + \frac{d\mu}{dx}\,y$$

$$\Rightarrow \quad \mu(x)\,P(x) = \frac{d\mu}{dx}$$

$$\Rightarrow \quad \frac{d\mu}{\mu} = P(x)\,dx$$

$$\Rightarrow \quad \ln\mu = \int P(x)\,dx$$

$$\Rightarrow \quad \boxed{\mu(x) = e^{\int P(x)\,dx}}$$

So: Method: ① Calculate $\mu(x) = e^{\int P(x)\,dx}$

② Write $\frac{d}{dx}\big(\mu(x)\,y\big) = Q(x)\,\mu(x)$

and integrate it.

Example $\quad \frac{dy}{dx} + \frac{2-3x^2}{x^3} y = 1 \quad \Rightarrow \quad P(x) = \frac{2}{x^3} - \frac{3}{x}$

$Q(x) = 1$

① $$\mu(x) = e^{\int\left(\frac{2}{x^3} - \frac{3}{x}\right) dx}$$
$$= e^{-\frac{1}{x^2} - 3\ln x}$$
$$= e^{-\frac{1}{x^2}} \cdot \frac{1}{x^3}$$

② so $$\frac{d}{dx}\left(\frac{1}{x^3} e^{-\frac{1}{x^2}} y\right) = \frac{1}{x^3} e^{-\frac{1}{x^2}} \cdot 1$$

integrate in x: (say from $x' = 1$ to x, to avoid singularity at $x=0$)

$$\frac{1}{x^3} e^{-\frac{1}{x^2}} y(x) - e^{-1} y(1) = \int_1^x \frac{1}{x'^3} e^{-\frac{1}{x'^2}} dx'$$
$$= \frac{1}{2}\left[e^{-\frac{1}{x'^2}}\right]_1^x$$
$$= \frac{1}{2}\left(e^{-\frac{1}{x^2}} - e^{-1}\right)$$

so
$$y(x) = e^{-1}\left(y(1) - \frac{1}{2}\right) x^3 e^{\frac{1}{x^2}} + \frac{x^3}{2}$$

3.2.4 Inexact equations

(A similar idea: see notes in RHB).

3.2.5 Homogeneous equations

Homogeneous equations are ODES that can be written as

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

They are suitably transformed into a simpler form by a change of variables:

$$v = \frac{y}{x} \quad (\text{or } y = vx) \quad \text{where } v = v(x)$$

Indeed:
$$\frac{dy}{dx} = \frac{d}{dx}(vx) = x\frac{dv}{dx} + v$$
$$= F(v)$$

So
$$x\frac{dv}{dx} + v = F(v)$$
$$\Rightarrow \frac{dv}{F(v)-v} = \frac{dx}{x}$$
which can be integrated in v and in x.

Example $(y-x)\frac{dy}{dx} + (2x+3y) = 0$

① Is it homogeneous? Yes: divide by x

$$\left(\frac{y}{x}-1\right)\frac{dy}{dx} + \left(2+\frac{3y}{x}\right) = 0$$

$$\frac{dy}{dx} + \frac{2+3\frac{y}{x}}{\frac{y}{x}-1} = 0$$

② Let $v = \frac{y}{x}$ then

$$x\frac{dv}{dx} + v + \frac{2+3v}{v-1} = 0$$

$$x\frac{dv}{dx} + \frac{(v-1)v + 2 + 3v}{v-1} = 0$$

$$x\frac{dv}{dx} + \frac{v^2+2v+2}{v-1} = 0 \rightarrow \underline{\text{Homework}}.$$

II Partial Differential equations (PDES)

① Mathematical definition

- A PDE is a functionnal relation between a function and its partial derivatives :

  let $f : \mathbb{R}^n \to \mathbb{R}$ then the most general form of a PDE is

$$\underbrace{\mathcal{F}}_{\text{a functionnal of}} \left(\underbrace{x_i}_{\text{the independent variables}} ; f ; \underbrace{\frac{\partial f}{\partial x_i}}_{\text{the first order partial derivatives}} ; \underbrace{\frac{\partial^2 f}{\partial x_i \partial x_j}}_{\text{the second order partial derivatives, etc...}} ; \dots \right) = 0 \qquad \begin{cases} i = 1 \dots n \\ j = 1 \dots n \end{cases}$$

- The order of the PDE is the order of the highest derivative involved

- A PDE is said to be linear if $\mathcal{F}$ is a linear combination of f and its derivatives :

$$a_0(x_1 \dots, x_n) f + a_1(x_1 \dots x_n) \frac{\partial f}{\partial x_1} + \dots + a_n(x_1 \dots x_n) \frac{\partial f}{\partial x_n}$$
$$+ a_{11}(x_1 \dots x_n) \frac{\partial^2 f}{\partial x_1^2} + \dots \quad a_{nn}(x_1 \dots x_n) \frac{\partial^2 f}{\partial x_n^2} + \dots$$
$$= b(x_1 \dots x_n)$$

- A linear PDE is homogeneous if $b = 0$

- A non linear PDE is not linear

<u>Examples</u> : $f_y + f_x = 0$ is linear, homogeneous, first-order

$f_{xx} + x f_y = 0$ is linear, second order, homogeneous

$3x f_x + f_{yy} = 4$ is linear, second order, non-homogeneous

$f_y + f_x f_{yy} = 0$ is non-linear.

- There exists <u>systems</u> of PDEs describing the coupled evolution of two or more dependent variables

<u>Example</u>:
$$\begin{cases} f_x + g_y f_{xx} = 0 \\ g_{xx} = 3 f_y \end{cases}$$
(a nonlinear coupled system of PDEs)

② <u>PDEs in real systems</u>

- Nature is classically represented by 3 spatial dimensions and one time dimension
  - ⇒ Most PDEs studied involve these 4 variables, or a subset of them

- The study of PDEs is often an attempt at modelling Nature
  - ⇒ There are usually 2 steps to the problem
    1. to construct a PDE which describes the problem (cf Applied Mathematics)
    2. to solve the PDE

- This class mostly focusses on ② although there are two aspects of ① worth describing here
  - notion of covariance
  - selection of a coordinate system

a. Notion of covariance

- The idea of a coordinate system is dependent on the observer / modeller. However, the physical process described by a PDE is usually independent of the observer.

  $\Rightarrow$ PDEs which describe a real phenomenon should first be written in a way that is independent of any coordinate system.

How? $\Rightarrow$ use universal differential operators:

e.g.: ∇f or $\nabla \cdot \underline{A}$ or $\nabla \times \underline{A}$

Example: The Laplace Equation: $\boxed{\nabla^2 f = 0}$

$\Rightarrow$ this is a universal equation, so that the solution to this equation, whether expressed in Cartesian coordinates (x, y, z)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

or in spherical polar coordinates (r, θ, ϕ)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial^2 f}{\partial \phi^2} = 0$$

represents the same scalar field f

- For a list of standard differential operators expressed in various coordinate systems see Handout.

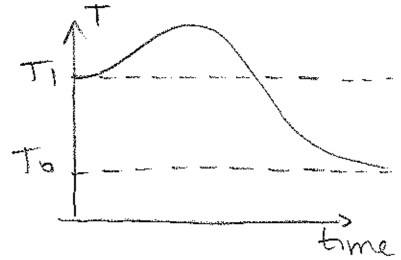
b. Selection of a coordinate system

For a given PDE, a judicious selection of a coordinate system is usually very useful. The idea is to use a system which best represents the symmetries of the physical problem

$\rightarrow$ to model a ball : use spherical coordinates
$\rightarrow$ to model a cylinder/cone : use cylindrical coordinates
$\rightarrow$ to model a cube/rectangle : use Cartesian coordinates

- Finally, it is <u>always</u> a good idea to use one's physical intuition of the system as a <u>guide</u> to find the solution or to critically assess the solution obtained.

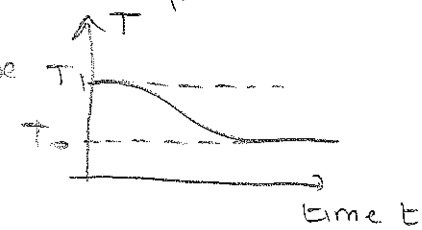
e.g. to model the cooling of a sphere initially at temperature T_1, immersed in a "bath" at temperature T_0 ⇒ we <u>know</u> that the sphere is unlikely to get hotter than T_1 at any time. ⇒ so the solution <u>cannot</u> be



⇐ NOT PHYSICAL

In fact, we also know that as $t \to \infty$, the temperature of the sphere should approach T_0

→ the solution is more likely to be



- In fact, we will see that for <u>LINEAR</u> PDEs there exists only 3 possible types of behaviour and these 3 are directly related to well-known physical systems

⇒ understanding the link between the <u>mathematical nature</u> of a PDE and its <u>physical behaviour</u> will help find solutions more easily

These are:
- the wave equation
- the heat equation (the diffusion equation)
- Laplace's equation

③ Fundamental 2nd order PDEs, (linear)

a. The wave equation

- Generally written as

$$\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f$$

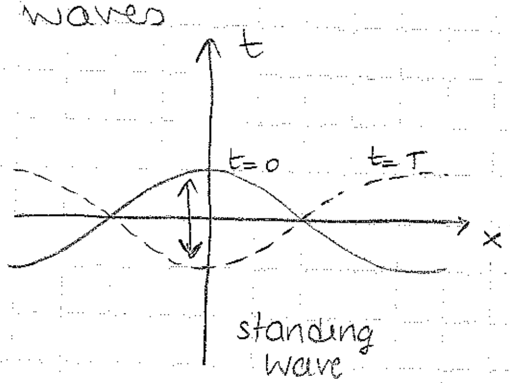
⇒ In cartesian coordinates for example

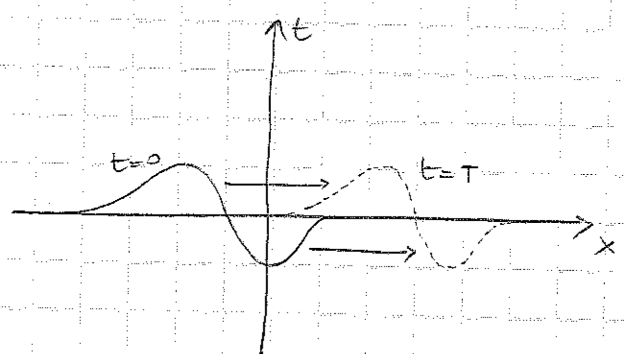
$$\frac{\partial^2 f}{\partial t^2} = c^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right)$$

- The typical behaviour of the solution is oscillatory in time & space

- There is no dissipation (energy is conserved)

- There can be standing waves or propagating waves





Examples in Nature
- sound waves
- light (electromagnetic waves)
- seismic waves
- water waves

⋮

10. The heat equation (diffusion equation)

- Generally written as

$$\frac{\partial f}{\partial t} = \nabla \cdot (k \nabla T) = k \nabla^2 T \quad \text{if } k \text{ is constant}$$

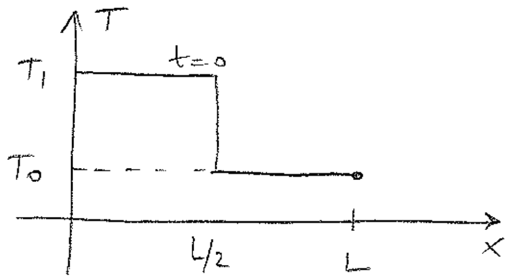
eg: in cartesian coordinates

$$\frac{\partial f}{\partial t} = k\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right)$$

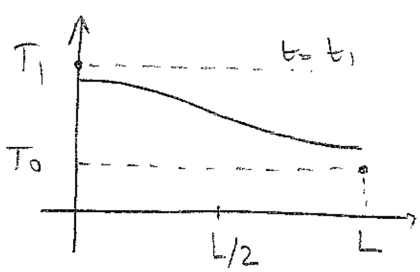
- This equation describes a diffusion process, its typical behaviour is to smooth out (and/or) <u>dissipate</u> gradients. The most commonly use example is the behaviour of a temperature field (hence the name "heat" equation).

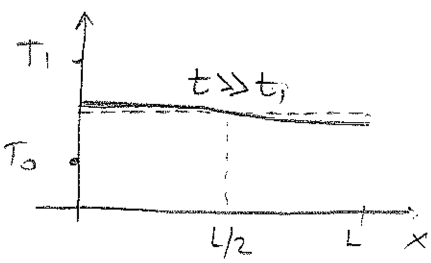
<u>Example</u>

If the initial condition for temperature in an insulated cylindrical thin rod of length L is



then





Physical intuition is <u>very</u> useful for diffusion problems.

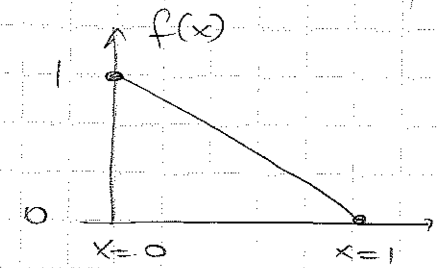
c. Laplace's equation

- Generally written as $\nabla^2 f = 0$

- Note that $\nabla^2 f = 0$ is the <u>steady-state</u> version of the heat equation $\Rightarrow$ can be thought of as the "end-product" of a diffusion process (after waiting an ∞ time)

<u>Example</u> = What is the solution to $\nabla^2 f = 0$ in 1D if $\begin{cases} f(x=0) = 1 \\ f(x=1) = 0 \end{cases}$

<u>Idea</u>. Imagine a 1-D rod of lenght 1, with one end held at temperature 1 and the other at temperature 0.
After an infinite time, the temperature in the rod has equilibrated to



$\rightarrow$ the solution to $\nabla^2 f = 0$ with these bcs.

<u>Check</u>. in 1D, $\nabla^2 f = f_{xx}$ so we simply solve

$f_{xx} = 0$ with above bcs

$\rightarrow$ solution is $f(x) = 1 - x$