NST PART 1A Mathematics III: course A **N13b**
Examples Sheet 2, Easter Term 2004 **H.E. Mason**

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1. Show that the functions $1, x, \frac{1}{2}(3x^2-1), \frac{1}{2}(5x^3-3x)$ are orthogonal on the interval $[-1,1]$.

2. Without resorting to integration, write down the Fourier series on $(-\pi,\pi)$ (with period 2π) for (i) $\sin 2\theta$ and (ii) $\cos^2\theta$. Obtain also the Fourier series for $\sin^3\theta$.

3. Say whether each one of the following functions of x are even, odd or neither:

$$\cos x, \quad \sin x, \quad \tan x, \quad \cos^2 x, \quad \sin^2 x, \quad x\cos x, \quad e^x, \quad \frac{(x-1)}{(x+1)}.$$

Given an arbitrary funtion $f(x)$, show that

$$F(x) = \frac{1}{2}(f(x)+f(-x)), \qquad G(x) = \frac{1}{2}(f(x)-f(-x)),$$

are respectively even and odd. Thus

$$f(x) = F(x) + G(x),$$

is the resolution of $f(x)$ into its even and odd parts. Perform this resolution for the functions in the previous list for which the answer neither was given.

4. A function $g(x)$ of period 2 is defined for $-1 < x < 1$ by $g(x) = x + |x|$, and by periodicity for all other x. Sketch this function, and without resorting to integation, instead using results given in lectures, *write down* its Fourier series.

5. Prove by integrating by parts that

$$\int_0^1 x^2\cos(n\pi x)dx = \frac{2(-)^n}{n^2\pi^2}.$$

6. An even function is defined by $f(x) = x^2$ for $-1 \le x \le 1$, and by periodicity elsewhere. Use the result of the previous question to show that it has the Fourier series

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty}\frac{4(-)^n}{n^2\pi^2}\cos(n\pi x).$$

Show (by putting $x = 1$) that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{6}\pi^2.$$

7. Let $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$ and $g(x) = \sum_{n=1}^{\infty} B_n \sin n\pi x$. show that

$$\int_{-1}^{1} f(x)g(x)dx = \sum_{n=1}^{\infty} b_n B_n.$$

What is the corresponding result when

$$f(x) = g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos n\pi x.$$

8. If $f(x) = g(x) = x^2$, combine the results of question six and the last result of question seven to deduce the result

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

9. (Harder question?) A function is defined by $f(x) = \cos x$ for $0 < x < \pi$, by $f(x) = -\cos x$ for $-\pi \le x \le 0$, and by periodicity elsewhere. Sketch this odd function carefully, and show that it has the Fourier sine series

$$f(x) = \sum_{n \text{ even}} \frac{4n}{\pi(n^2-1)} \sin nx = \sum_{r=1}^{\infty} \frac{8r}{\pi(4r^2-1)} \sin 2rx.$$

How do you interpret this result for $x = 0$?

(This question has found a half-range sine series for the function $f(x) = \cos x$ defined for $0 \le x \le 1$.)