$\boxed{6.1}$ $\quad u'' + \lambda u = 0 \qquad 0 < x < 1$

$u(0) = u'(0) \quad \rightarrow \quad p(x) = 1 \quad q(x) = 0 \quad r(x) = 1$

$u(1) = -u'(1)$

(a)
$$R(u) = \frac{\int_0^1 u'^2 dx - [uu']_0^1}{\int_0^1 u^2 dx}$$

$$= \frac{\int_0^1 u'^2 dx + u(1)^2 + u(0)^2}{\int_0^1 u^2 dx} \geq 0$$

so $\lambda_n \geq 0 \ \forall n$.

(b)
$$u = \alpha \cos\sqrt{\lambda}x + \beta \sin\sqrt{\lambda}x$$
$$u' = -\sqrt{\lambda}\alpha \sin\sqrt{\lambda}x + \beta\sqrt{\lambda}\cos\sqrt{\lambda}x$$
$$u(0) = u'(0) \implies \alpha = \beta\sqrt{\lambda}$$
$$u(1) = -u'(1) \implies \beta\sqrt{\lambda}\cos\sqrt{\lambda} + \beta\sin\sqrt{\lambda} = \beta\lambda\sin\sqrt{\lambda} - \beta\sqrt{\lambda}\cos\sqrt{\lambda}$$
$$\implies 2\cancel{\beta}\sqrt{\lambda}\cos\sqrt{\lambda} = \cancel{\beta}(\lambda - 1)\sin\sqrt{\lambda}$$
$$\implies \frac{2\sqrt{\lambda}}{\lambda - 1} = \tan\sqrt{\lambda}$$

Let the solutions of this equation be λ_n then

$$\alpha_n = \beta\sqrt{\lambda_n}$$

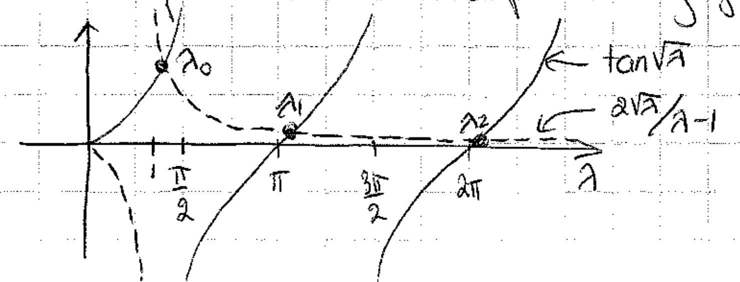
and

$$u_n = \beta_n\sqrt{\lambda_n}\cos\sqrt{\lambda_n}x + \beta_n\sin\sqrt{\lambda_n}x$$

(c) For large eigenvalues

$$\lambda_n \simeq \left(\frac{n\pi}{\int_0^1 \sqrt{\frac{r(x)}{p(x)}}dx}\right)^2 \simeq \left(\frac{n\pi}{\int_0^1 dx}\right)^2 \simeq n^2\pi^2$$

Note: this is expected from the following graph:



$\boxed{6.2}$ $(xu')' + \frac{\lambda}{x}u = 0$ $x \in [1, e]$

$u(1) = u(e) = 0$

- Try a solution of the kind

$$u(x) = x^{\alpha} \quad \text{then}$$

$$(xu')' + \frac{\lambda}{x}u = 0 \iff \alpha^2 + \lambda = 0$$

$$\iff \alpha = \pm\sqrt{\lambda}$$

We suspect that $\lambda > 0 \Rightarrow \alpha = \pm i\sqrt{\lambda}$

So $$u(x) = A x^{i\sqrt{\lambda}} + B x^{-i\sqrt{\lambda}}$$

$$= A e^{i\sqrt{\lambda}\ln x} + B e^{-i\sqrt{\lambda}\ln x}$$

$$= a\cos(\sqrt{\lambda}\ln x) + b\sin(\sqrt{\lambda}\ln x)$$

To fit the bcs we require

$$u'(x) = a\frac{\sqrt{\lambda}}{x}\left(-\sin(\sqrt{\lambda}\ln x)\right) + \frac{\sqrt{\lambda}}{x}b\cos(\sqrt{\lambda}\ln x)$$

$u(1) = 0 \Rightarrow a = 0$

$u'(e) = 0 \Rightarrow b\frac{\sqrt{\lambda}}{e}\cos(\sqrt{\lambda}) = 0 \Rightarrow \sqrt{\lambda} = \frac{\pi}{2} + n\pi$

So $$u_n(x) = b_n \left(\frac{\pi}{2} + n\pi\right) \sin\left[\left(\frac{\pi}{2} + n\pi\right)\ln x\right]$$

- $\langle u_n(x), u_m(x)\rangle = \int_{10}^{e} u_n(x)u_m(x)r(x)\,dx$ with $r(x) = \frac{1}{x}$

$$\langle u_n(x), u_m(x)\rangle = \int_1^e \frac{b_n b_m}{x}\left(\frac{\pi}{2}+n\pi\right)\left(\frac{\pi}{2}+m\pi\right)\sin\left[\left(\frac{\pi}{2}+n\pi\right)\ln x\right] \sin\left[\left(\frac{\pi}{2}+m\pi\right)\ln x\right] dx$$

let $\xi = \ln x$ so

$d\xi = \frac{1}{x} dx$ then

$$\langle u_n(x), u_m(x) \rangle = \int_0^1 b_n b_m \left(\frac{\pi}{2} + n\pi\right)\left(\frac{\pi}{2} + m\pi\right) \sin\left[\left(\frac{\pi}{2} + n\pi\right)\xi\right] \sin\left[\left(\frac{\pi}{2} + m\pi\right)\xi\right] d\xi$$

$$\sin a \sin b = \frac{\cos(a-b)}{2} - \frac{\cos(a+b)}{2}$$

$$\langle u_n, u_m \rangle = \int_0^1 \frac{b_n b_m}{2} \left(\frac{\pi}{2} + n\pi\right)\left(\frac{\pi}{2} + m\pi\right)\left[\cos\left((n-m)\pi\xi\right) - \cos\left((n+m+1)\pi\xi\right)\right] d\xi$$

$= 0$ unless $n = m$.

$\boxed{6.3}$ $\qquad (x^2 u')' + \lambda u = 0 \qquad\qquad x \in [1, b]$

$$u(1) = u(b) = 0$$

- Try a solution of the kind $u = x^\alpha$ then

$$\alpha(\alpha+1) + \lambda = 0 \qquad \alpha^2 + \alpha + \lambda = 0$$

$$\Rightarrow \qquad \alpha = \frac{-1 \pm \sqrt{1 - 4\lambda}}{2}$$

We suspect that $1 - 4\lambda < 0$ so

$$\alpha = \frac{-1 \pm i\sqrt{4\lambda - 1}}{2}$$

So $\quad u = Ax^{-\frac{1}{2} + i\frac{\sqrt{4\lambda-1}}{2}} + Bx^{-\frac{1}{2} - i\frac{\sqrt{4\lambda-1}}{2}}$

$$= Ae^{\left(-\frac{1}{2} + \frac{i}{2}\sqrt{4\lambda-1}\right)\ln x} + Be^{\left(-\frac{1}{2} - \frac{i}{2}\sqrt{4\lambda-1}\right)\ln x}$$

$$= x^{-\frac{1}{2}}\left(a\cos\left(\frac{\sqrt{4\lambda-1}}{2}\ln x\right) + \beta\sin\left(\frac{\sqrt{4\lambda-1}}{2}\ln x\right)\right)$$

$$u(1) = 0 \quad \Rightarrow \quad a = 0$$

$$u(b) = 0 \quad \Rightarrow \quad b^{-\frac{1}{2}}\beta\sin\left[\frac{\sqrt{4\lambda-1}}{2}\ln b\right] = 0$$

$$\Rightarrow \quad \frac{\sqrt{4\lambda_n - 1}}{2}\ln b = n\pi$$

$$4\lambda_n - 1 = \frac{4n^2\pi^2}{(\ln b)^2}$$

$$\Rightarrow \lambda_n = \frac{n^2\pi^2}{(\ln b)^2} + \frac{1}{4}$$

and

$$u_n(x) = \beta_n x^{-\frac{1}{2}} \sin\left(\frac{n\pi}{\ln b}\ln x\right)$$

• Given $u_t = (x^2 u_x)_x \qquad 1 < x < b\ ,\ t > 0$

$$u(1,t) = u(b,t) = 0$$

$$u(x,0) = f(x)$$

① Separate the variables: let $u(x,t) = A(x)B(t)$

$$\Rightarrow \quad \dot{B} = -\lambda B$$

$$(x^2 A_x)_x + \lambda A = 0$$

we know the solutions to the 2nd equation;

So $\quad u(x,t) = \sum_n \beta_n e^{-\lambda_n t} x^{-\frac{1}{2}} \sin\left(\frac{n\pi}{\ln b} \ln x\right)$

with $\quad \lambda_n = \frac{n^2\pi^2}{(\ln b)^2} + \frac{1}{4}$

When $t=0$ we need to fit

$$u(x,0) = \sum_n \beta_n x^{-\frac{1}{2}} \sin\left(\frac{\pi n}{\ln b} \ln x\right) = f(x)$$

$$\Rightarrow \quad \sum_n \beta_n \sin\left(\frac{n\pi}{\ln b} \ln x\right) = x^{\frac{1}{2}} f(x) \qquad 1 < x < b$$

let $u = \frac{\ln x}{\ln b} \quad (u \in [0,1])$ then $x = e^{u \ln b} = b^u$

So $\quad \sum_n \beta_n \sin(n\pi u) = \qquad b^{\frac{u}{2}} f(b^u) = F(u)$

F is defined between 0 and 1; we need to construct the 2-periodic odd function in $[-1,1]$ G such that $\quad G(u) = F(u) \quad x \in [0,1]$

$\qquad G(u) = -F(u) \quad x \in [-1,0]$

Then

$$\beta_n = 2\int_0^1 \sin(n\pi u)\, F(u)\, du .$$

6.5

$$\begin{cases} ((1+x)^2 u')' + \lambda u = 0 \\ u(0) = u(1) = 0 \end{cases}$$

Preliminary

here $p(x) = (1+x)^2$ and the problem is regular.

$r(x) = 1$

$q(x) = 0$

So $$R(u) = \frac{\int_0^1 (1+x)^2 u'^2\, dx}{\int_0^1 u^2\, dx} \Rightarrow R(u) \geq 0 \Rightarrow \lambda_0 \geq 0$$

Let's try $u = x(1-x)$ then $u' = 1-2x$

$$R(u) = \frac{\int_0^1 (1+x)^2 (1-2x)^2\, dx}{\int_0^1 x^2(1-x)^2\, dx} = \frac{\int_0^1 (1-10x+5x^2+4x^3+4x^4)\,dx}{\int_0^1 (x^2-2x^4+x^6)\, dx}$$

$$\frac{1-5+\frac{5}{3}+1+\frac{4}{5}}{\frac{1}{3}-\frac{2}{5}+\frac{1}{7}} =$$

① How can we solve it?

First notice that if $y = 1+x$ then $\frac{du}{dx} = \frac{dy}{dy} \Rightarrow$

$$\begin{cases} (y^2 u')' + \lambda u = 0 \\ u(1) = u(2) = 0 \end{cases}$$ in terms of the indep. variable $y \in [1,2]$.

Now notice that $u = y^\alpha$ is a solution provided

$$\alpha(\alpha-1) + 2\alpha + \lambda = 0$$

$$\Leftrightarrow \quad \alpha^2 + \alpha + \lambda = 0$$

$$\alpha = \frac{-1 \pm \sqrt{1-4\lambda}}{2}$$

So if $\lambda < \frac{1}{4}$ $\quad u = Ay^{\alpha_+} + By^{\alpha_-}$

with α_+, α_- real which would imply

(using the BCs) that

$$\begin{cases} A + B = 0 \\ A2^{\alpha_+} + B2^{\alpha_-} = 0 \end{cases} \Rightarrow \text{impossible}$$

So we deduce $\lambda > \frac{1}{4}$ then

$$u = Ae^{\alpha_+ \ln y} + Be^{\alpha_- \ln y}$$

where $\alpha_+ = \dfrac{-1 + i\sqrt{4\lambda - 1}}{2}$ $\qquad \alpha_- = \dfrac{-1 - i\sqrt{4\lambda-1}}{2}$

So $$u = Ae^{-\frac{1}{2}\ln y + i\sqrt{\lambda - 1/4}\ln y} + Be^{-\frac{1}{2}\ln y - i\sqrt{\lambda - 1/4}\ln y}$$

$$= y^{-\frac{1}{2}}\left[Ae^{i\sqrt{\lambda - \frac{1}{4}}\ln y} + Be^{-\sqrt{\lambda - 1/4}\ln y}\right]$$

$$= y^{-\frac{1}{2}}\left[a\cos\left[\sqrt{\lambda - \tfrac{1}{4}}\ln y\right] + b\sin\left[\sqrt{\lambda - \tfrac{1}{4}}\ln y\right]\right]$$

since $u(1) = 0$ then $a = 0$

$u(2) = 0$ then $\quad \sin\left[\sqrt{\lambda_n - \frac{1}{4}}\ln 2\right] = 0$

$$\Rightarrow \sqrt{\lambda_n - \tfrac{1}{4}}\ln 2 = n\pi$$

$$\Rightarrow \lambda_n = \frac{1}{4} + \frac{n^2\pi^2}{(\ln 2)^2}$$

And we can now write

$$u_n(x) = \frac{b}{\sqrt{1+x}}\sin\left(\frac{n\pi}{\ln 2}\ln(1+x)\right) \qquad n \geq 1.$$

② Orthogonality: the inner product considered is

$$\int_0^1 u_n(x)\, u_m(x)\, dx$$

$\Rightarrow$ we want to show

$$\int_0^1 \frac{1}{\sqrt{1+x}} \sin\left(\frac{n\pi}{\ln 2}\ln(1+x)\right)\frac{1}{\sqrt{1+x}}\sin\left(\frac{m\pi}{\ln 2}\ln(1+x)\right) dx = 0$$

when $n \neq m$

$$\Leftrightarrow \int_0^1 \frac{1}{1+x} \sin\left(\frac{n\pi}{\ln 2}\ln(1+x)\right)\sin\left(\frac{m\pi}{\ln 2}\ln(1+x)\right)dx$$

Let $y = \ln(1+x)$ then

$$dy = \frac{dx}{1+x}$$

$$\Rightarrow \int_0^{\ln 2} \sin\left(\frac{n\pi}{\ln 2}y\right)\sin\left(\frac{m\pi}{\ln 2}y\right)dy.$$

$$= \frac{1}{2}\int_0^{\ln 2} \cos\left[\frac{(n-m)\pi y}{\ln 2}\right] - \cos\left[\frac{(m+n)\pi y}{\ln 2}\right] dy$$

$$= \frac{1}{2}\left\{\frac{\ln 2}{(n-m)\pi}\sin\left(\frac{(n-m)\pi y}{\ln 2}\right) - \frac{\ln 2}{(m+n)\pi}\sin\left(\frac{(m+n)\pi y}{\ln 2}\right)\right\}_0^{\ln 2}$$

OK since $n \neq m$

$$= \frac{1}{2}\{0\}$$

$\boxed{6.4}$

$$u'' + (\lambda - x^2)u = 0 \qquad \to \text{see lecture notes}$$
$$u'(0) = u(1) = 0$$

$\boxed{6.6}$ $\quad u'' + (\lambda - x^2)u = 0 \quad$ Prove that all $\lambda > 0$.
$$u'(0) = u'(1) = 0$$

$\to$ Calculate the Rayleigh quotient:

$$R(u) = -\frac{\int_0^1 u L(u)\,dx}{\int_0^1 r u^2\,dx}$$

with $L(u) = u'' - x^2 u \qquad r(x) = 1$
$$(p(x) = 1,\ q(x) = -x^2).$$

$$R(u) = -\frac{\int_0^1 uu'' - x^2u^2\,dx}{\int_0^1 u^2\,dx}$$

$$= -\frac{[uu']_0^1 - \int_0^1 (u'^2 + x^2u^2)\,dx}{\int_0^1 u^2\,dx}$$

use $u'(0) = u'(1) = 0$

$$= \frac{\int_0^1 (u'^2 + x^2u^2)\,dx}{\int_0^1 u^2\,dx} > 0 \text{ unless } u \text{ is identically } 0.$$

$$\to \boxed{\lambda > 0}$$

$\boxed{6.21}$ $u_t - u_{xx} = t\cos(2001x)$

$u_x(0,t) = u_x(1,t) = 0 \qquad t \geq 0$

$u(x,0) = \pi \cos 2x$.

1. Seek eigenmodes of spatial, homogeneous problem

$$B_{xx} = +\lambda B$$

Homogeneous bcs $\Rightarrow$ oscillatory solutions needed here so $\lambda = -k^2$

$$B(x) = \alpha\cos(kx) + \beta\sin kx$$
$$\Rightarrow B_x = -\alpha k\sin(kx) + \beta k\cos kx.$$

To fit the bcs we need $\begin{cases}\sin(k\pi) = 0 \Rightarrow k = n \\ \beta = 0\end{cases}$

So $\boxed{B_n(x) = \cos nx.}$

2. Assume solution is $u(x,t) = \sum_{n=0}^{\infty} A_n(t) B_n(x)$

$$\Rightarrow \sum_{n=0}^{\infty} \dot{A}_n(t)\cos nx + n^2 A_n(t)\cos nx = t\cos 2001x$$

Project onto eigenmodes using

$$\int_0^1 \cos nx \cos mx \, dx = \begin{cases} \frac{\pi}{2} & \text{if } m = n \neq 0 \\ \pi & \text{if } m = n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \begin{cases} m = 0: & \dot{A}_0(t) = 0 \\ m > 0: & \dot{A}_m(t) + m^2 A_m(t) = t\,\delta_{m,2001}. \end{cases}$$

$$\Rightarrow \quad A_0(t) = a_0$$

$$A_m(t) = a_n e^{-m^2 t} + (P.S.) \cdot \delta_{m,2001}$$

$m = 2001$: To find the particular solution, try a linear function

$$ct + d$$

$$\rightarrow \quad c + (2001)^2 (ct + d) = t$$

$$\Rightarrow \quad c = \frac{1}{(2001)^2} \qquad d = -\frac{1}{(2001)^4}$$

so

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos nx + \frac{1}{(2001)^2}\left(t - \frac{1}{(2001)^2}\right) \cdot \cos(2001x).$$

3. To find a_n: apply initial condition.

$$u(x,0) = \pi \cos 2x = a_0 + \sum_{n=1}^{\infty} a_n \cos nx - \frac{1}{(2001)^4} \cos(2001x)$$

by looking at coeffs:

$$a_0 = 0$$

$$a_2 = \pi$$

$$a_{2001} = \frac{1}{(2001)^4}$$

so finally

$$u(x,t) = \pi \cos(2x) e^{-4t} + \frac{1}{(2001)^4} \cos(2001x) e^{-(2001)^2 t}$$

$$+ \frac{1}{(2001)^2}\left(t - \frac{1}{(2001)^2}\right) \cos(2001x).$$

$\boxed{6-24}$

$$u_{tt} - u_{xx} = \cos 2t \cos 3x$$

$$u_x(0,t) = u_x(\pi,t) = 0$$

$$u(x,0) = \cos^2 x$$

$$u_t(x,0) = 1$$

1. Solve for the eigenmodes of the spatial homogeneous problem:

$$B_{xx} = \lambda B$$

↳ Since the bcs are homogeneous too, we need oscillatory functions ($\Rightarrow \lambda < 0$, let $\lambda = -k^2$)

↳ see 6-21, $B_n = \cos nx$

2. Try $u(x,t) = \sum_{n=0}^{\infty} A_n(t) B_n(x)$

$$\Rightarrow \sum_{n=0}^{\infty} \ddot{A}_n(t) \cos nx + n^2 A_n(t) \cos nx = \cos 2t \cos 3x$$

Use orthogonality to get

- $m = 0$: $\ddot{A}_0 = 0$
- $m > 0$ $\ddot{A}_m(t) + m^2 A_m(t) = \cos 2t \; \delta_{m,3}$

↳ Solutions: $A_0(t) = a_0 t + b_0$

$$A_m(t) = a_m \cos mt + b_m \sin mt + (P.S.)\, \delta_{m,3}$$

$m=3$: To find the particular solution, try $K \cos 2t$

$$\Rightarrow -4K + 9K = 1 \Rightarrow K = \frac{1}{5}$$

So

$$u(x,t) = a_0 t + b_0 + \sum_{m=1}^{\infty} \left(a_m \cos mt + b_m \sin mt \right) \cos mx$$

$$+ \frac{1}{5} \cos 2t \cos 3x$$

3. Apply initial conditions:

<u>at $t=0$</u>: $u(x,0) = \cos^2 x = b_0 + \sum_{m=1}^{\infty} a_m \cos mx$

$$+ \frac{1}{5} \cos 3x$$

so since $\cos^2 x = \dfrac{1+\cos 2x}{2}$ we can identify

$$b_0 = \frac{1}{2}$$

$$a_2 = \frac{1}{2}$$

$$a_3 + \frac{1}{5} = 0 \Rightarrow a_3 = -\frac{1}{5}$$

$$u_t(x,0) = 1 = a_0 + \sum_{m=1}^{\infty} m b_m \cos mx$$

$$\Rightarrow \begin{cases} a_0 = 1 \\ b_m = 0 \end{cases}$$

<u>So</u>:

$$u(x,t) = t + \frac{1}{2} + \frac{1}{2} \cos 2t \cos 2x - \frac{1}{5} \cos 3t \cos 3x$$
$$+ \frac{1}{5} \cos 2t \cos 3x$$