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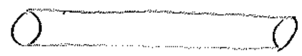

$x=0$ $\qquad$ $x=10$

$$T(x,0) = 100$$
$$T(0,t) = T(10,t) = 0 \quad \forall t>0$$

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$$

Separation of variables: $T(x,t) = A(x)B(t)$

$$\Rightarrow \quad k\frac{A_{xx}}{A} = \frac{B_t}{B} = \text{constant}$$

- if constant is >0 $\Rightarrow$ exponentially growing solutions, unphysical
- if constant $=0$ $\Rightarrow$ picks up steady-state solution solution of $A_{xx}=0$ with $A(0)=A(10)=0$
  $\rightarrow$ solution is $A_0 = 0$
- if constant is <0 $\Rightarrow$ let $\frac{1}{k}\frac{B_t}{B} = \frac{A_{xx}}{A} = -\lambda^2$

then
$$\begin{cases} A(x) = a\cos\lambda x + b\sin\lambda x \\ B(t) = e^{-\lambda^2 k t} \end{cases}$$

$$A(0) = A(10) = 0 \Rightarrow a=0 \text{ and } \lambda_n = \frac{n\pi}{10}$$

$\Rightarrow$ General solution:

$$T(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right) e^{-\frac{n^2\pi^2}{100}kt}$$

- at $t=0$ $\qquad 100 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right)$

with
$$b_n = \frac{2}{10}\int_0^{10} 100 \sin\left(\frac{n\pi x}{10}\right)$$
$$= 20\,\frac{10}{n\pi}\left[-\cos\frac{n\pi x}{10}\right]_0^{10}$$
$$= \frac{200}{n\pi}(1-\cos n\pi) = \frac{400}{n\pi} \text{ if } n \text{ odd.}$$

$$\Rightarrow \quad T(x,t) = \frac{400}{\pi}\sum_{\substack{\text{odd}\\ n}} \sin\left(\frac{n\pi x}{10}\right) e^{-\frac{n^2\pi^2}{100}kt}$$

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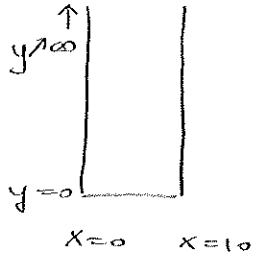

$$T(x,0) = x$$
$$T(0,y) = T(10,y) = 0$$
$$\nabla^2 T = 0$$

- Separation of variables. $T(x,y) = A(x)B(y)$

$$\Rightarrow \quad \frac{A_{xx}}{A} + \frac{B_{yy}}{B} = 0 \quad \Rightarrow \quad \frac{A_{xx}}{A} = -\frac{B_{yy}}{B} = \text{constant}$$

- To fit the boundary conditions @ $x=0$ and $x=10$, we see that $A(x)$ has to be a linear combination of sin & cos function and not a linear combination of exponentials $\Rightarrow$ the constant must be negative so let

$$A_{xx} = -K^2 A \quad \rightarrow \quad A(x) = a\cos(kx) + b\sin(kx)$$
$$B_{yy} = K^2 B \quad \rightarrow \quad B(y) = \alpha e^{kx} + \beta e^{-kx}$$

- $A(0) = 0 \Rightarrow a = 0$

$A(10) = 0 \Rightarrow K_n = \frac{n\pi}{10}$ so $A_n(x) = b_n \sin\left(\frac{n\pi x}{10}\right)$

$B(\infty) < +\infty \Rightarrow \alpha = 0$ so $B_n(y) = e^{-\frac{n\pi}{10}y}$

- The general solution is therefore $T(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right) e^{-\frac{n\pi}{10}y}$

- $T(x,0) = x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right)$

with
$$b_n = \frac{2}{10}\int_0^{10} x \sin\left(\frac{n\pi x}{10}\right)$$
$$= \frac{2}{10}\left[-\frac{10}{n\pi} x \cos\left(\frac{n\pi x}{10}\right)\right]_0^{10} + \frac{2}{10}\int_0^{10} \frac{10}{n\pi}\cos\left(\frac{n\pi x}{10}\right)dx$$
$$= -\frac{20}{n\pi}\cos(n\pi) = \frac{20}{n\pi}(-1)^{n+1}$$

as required ✓

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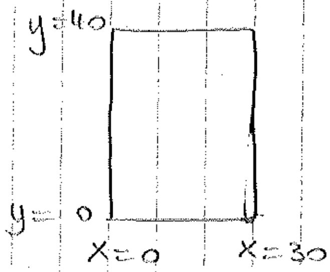

$$T(0,y) = T(30,y) = 0$$

$$T(x,0) = 0$$

$$T(x,40) = \begin{cases} 100 & x \in [0,10] \\ 0 & x \in [10,30] \end{cases}$$

- Separation of variables $T(x,y) = A(x)B(y)$

$$\Rightarrow \frac{A_{xx}}{A} + \frac{B_{yy}}{B} = 0 \Rightarrow \frac{A_{xx}}{A} = -\frac{B_{yy}}{B} = \text{constant}$$

- To fit homogeneous boundary conditions in x, we must have oscillatory functions in x $\Rightarrow$ the constant must be negative $\Rightarrow$ let

$$\begin{cases} \frac{A_{xx}}{A} = -k^2 & A(x) = a\cos kx + b\sin kx \\ \frac{B_{yy}}{B} = k^2 & B(y) = \alpha\cosh(ky) + \beta\sinh(ky) \end{cases}$$

- $A(0) = 0 \Rightarrow a = 0$ so $A_n(x) = b_n \sin\left(\frac{n\pi x}{30}\right)$
  $A(30) = 0 \Rightarrow k = \frac{n\pi}{30}$

  $B(0) = 0 \Rightarrow \alpha = 0$ so $B_n(y) = \sinh\left(\frac{n\pi y}{30}\right)$

$\Rightarrow$ General solution: $T(x,y) = \sum_1 b_n \sin\left(\frac{n\pi x}{30}\right)\sinh\left(\frac{n\pi y}{30}\right)$

- At $y = 40$ $\quad T(x,y) = \sum_1 b_n \sinh\left(\frac{4n\pi}{3}\right)\sin\left(\frac{n\pi x}{30}\right) = \begin{cases} 100 & x \in [0,10] \\ 0 & x \in [10,30] \end{cases}$

$$\sinh\left(\frac{4n\pi}{3}\right) b_n = \frac{2}{30}\int_0^{30} \sin\left(\frac{n\pi x}{30}\right) T(x,40)\,dx$$

$$= \frac{2}{30}\int_0^{10} 100 \sin\left(\frac{n\pi x}{30}\right) = \frac{20}{3}\cdot\frac{30}{n\pi}\left[-\cos\frac{n\pi x}{30}\right]_0^{10}$$

$$= \frac{200}{n\pi}\left(1 - \cos\frac{n\pi}{3}\right)$$

$$\Rightarrow T(x,y) = \sum_1 \frac{200}{n\pi}\left(1 - \cos\frac{n\pi}{3}\right)\sin\left(\frac{n\pi x}{30}\right)\frac{\sinh\left(\frac{n\pi y}{30}\right)}{\sinh(4n\pi/3)}$$

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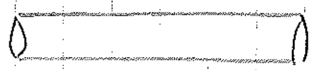

$T(x,0) = x$

$\frac{\partial T}{\partial x}\Big|_0 = \frac{\partial T}{\partial x}\Big|_l = 0$

$\frac{\partial T}{\partial t} = k\nabla^2 T$

- Separation of variables $T(x,t) = A(x)B(t)$
- $\Rightarrow \quad \frac{B_t}{B} = k\frac{A_{xx}}{A} = \text{constant}$
- The constant cannot be >0, otherwise exponentially growing solutions, unphysical
- If constant $=0$ then $A_{xx}=0$ with $\frac{\partial A}{\partial x}\Big|_0 = \frac{\partial A}{\partial x}\Big|_l = 0$

  $\Rightarrow \quad A_0(x) = a_0$
- If constant <0 then let $\frac{1}{k}\frac{B_t}{B} = \frac{A_{xx}}{A} = -\lambda^2$

$$\Rightarrow \begin{cases} A(x) = a\cos\lambda x + b\sin\lambda x \\ B(t) = e^{-\lambda^2 kt} \end{cases}$$

From $\frac{dA}{dx}\Big|_0 = 0 \Rightarrow b = 0$

$\frac{dA}{dx}\Big|_l = 0 \Rightarrow \lambda = \frac{n\pi}{l}$

$\Rightarrow$ General solution: $T(x,t) = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right)e^{-n^2\pi^2 kt}$

- At $t=0$ $T(x,0) = x = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right)$

with $a_0 = \frac{1}{l}\int_0^l x\,dx = \frac{l}{2}$

$$a_n = \frac{2}{l}\int_0^l x\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\left[\frac{l}{n\pi}x\sin\left(\frac{n\pi x}{l}\right)\right]_0^l - \frac{2}{n\pi}\int_0^l \sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2l^2}{n^2\pi^2}\left(\cos n\pi - 1\right) = -\frac{4l^2}{n^2\pi^2}$$ for n odd

as required

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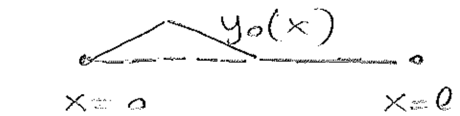

$$y_{tt} = c^2 y_{xx}$$
$$y(0,t) = y(\ell,t) = 0$$
$$y(x,0) = y_0(x)$$
$$y_t(x,0) = 0$$

Separation of variables : $y(x,t) = A(x)B(t)$

$$\Rightarrow \quad \frac{B_{tt}}{B} = c^2 \frac{A_{xx}}{A} = \text{constant}$$

We expect oscillatory behavior in time $\Rightarrow$ constant is negative

$$\rightarrow \text{let} \quad \frac{1}{c^2}\frac{B_{tt}}{B} = \frac{A_{xx}}{A} = -\lambda^2$$

$$\Rightarrow \begin{cases} A(x) = a\cos\lambda x + b\sin\lambda x \\ B(t) = \alpha\cos c\lambda t + \beta\sin c\lambda t \end{cases}$$

$$A(0) = A(\ell) = 0 \Rightarrow a = 0 \text{ and } \lambda = \frac{n\pi}{\ell}$$

$\Rightarrow$ General solution is

$$y(x,t) = \sum_{n=1} \sin\left(\frac{n\pi x}{\ell}\right)\left[\alpha_n\cos\left(\frac{n\pi ct}{\ell}\right) + \beta_n\sin\left(\frac{n\pi ct}{\ell}\right)\right]$$

- $y_t(x,0) = 0 \Rightarrow \beta_n = 0 \quad \forall n$
- $y(x,0) = y_0(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{\ell}\right)\alpha_n$

$$\Rightarrow \alpha_n = \frac{2}{\ell}\int_0^{\ell} y_0(x)\sin\left(\frac{n\pi x}{\ell}\right)dx = \frac{2}{\ell}\int_0^{\frac{\ell}{2}} y_0(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$= \frac{2}{\ell}\int_0^{\ell/4} \frac{hx}{\frac{\ell}{4}}\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$+ \frac{2}{\ell}\int_{\ell/4}^{\ell/2}\left(2h - \frac{hx}{\frac{\ell}{4}}\right)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

But
$$\int_a^b x\sin\lambda x\,dx = \frac{1}{\lambda}\left[a\cos\lambda a - b\cos\lambda b\right] + \frac{1}{\lambda^2}\left(\sin\lambda b - \sin\lambda a\right)$$

So $\alpha_n = \frac{8}{\ell^2} h \left\{ \left[\frac{\ell}{n\pi} \right] \left[-\frac{\ell}{4} \cos\left(\frac{n\pi}{\ell} \cdot \frac{\ell}{4}\right) \right] + \frac{\ell^2}{n^2\pi^2} \sin\left(\frac{n\pi}{\ell}\frac{\ell}{4}\right) \right\}$

$$+ \frac{4h}{\ell} \frac{\ell}{n\pi} \left[\cos \frac{n\pi}{\ell}\frac{\ell}{4} - \cos \frac{n\pi}{\ell}\frac{\ell}{2} \right]$$

$$- \frac{8h}{\ell^2} \left\{ \frac{\ell}{n\pi} \left[\frac{\ell}{4} \cos \frac{n\pi}{\ell}\frac{\ell}{4} - \frac{\ell}{2} \cos \frac{n\pi}{2}\frac{\ell}{2} \right] + \frac{\ell^2}{n^2\pi^2} \left(\sin \frac{n\pi}{\ell}\frac{\ell}{2} - \sin \frac{n\pi}{\ell}\frac{\ell}{4} \right) \right\}$$

$$= \frac{16h}{n^2\pi^2} \sin\left(\frac{n\pi}{4}\right) - \frac{8h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

as required

$\boxed{\text{Problem 5 p 558}}$

String with $y_{tt} = c^2 y_{xx}$

$y(0,t) = y(\ell,t) = 0$

$y(x,0) = 0$

$$y_t(x,0) = \begin{cases} \frac{hx}{\ell/2} & \text{if } x \in [0, \ell/2] \\ 2h - \frac{hx}{\ell/2} & \text{if } x \in [\ell/2, \ell] \end{cases}$$

The start is the same as the previous problem

$\Rightarrow$ general solution is

$$y(x,t) = \sum_1 \sin\left(\frac{n\pi x}{\ell}\right)\left(\alpha_n \cos\left(\frac{n\pi c t}{\ell}\right) + \beta_n \sin\left(\frac{n\pi c t}{\ell}\right)\right)$$

- $y(x,0) = 0 \Rightarrow \boxed{\alpha_n = 0}$

$y_t(x,0) = f(x) \Rightarrow \sum_1 \beta_n \frac{n\pi c}{\ell} \sin\left(\frac{n\pi x}{\ell}\right) = f(x)$

so $\quad \frac{n\pi c}{\ell}\beta_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$

$$= \frac{2}{\ell}\int_0^{\ell/2}\frac{hx}{\ell/2}\sin\left(\frac{n\pi x}{\ell}\right)dx + \frac{2}{\ell}\int_{\ell/2}^{\ell}\left(2h - \frac{hx}{\ell/2}\right)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$= \frac{4h}{\ell^2}\left\{\frac{\ell}{n\pi}\left(\cancel{-\frac{\ell}{2}\cos\frac{n\pi}{\ell}\cdot\frac{\ell}{2}}\right)\right.$$

$$\left. + \frac{\ell^2}{n^2\pi^2}\left(\sin\frac{n\pi}{\ell}\frac{\ell}{2}\right)\right\}$$

$$+ \frac{2}{\ell}\cdot 2h\cdot\left[\frac{\ell}{n\pi}\right]\left(\cancel{\cos\frac{n\pi}{\ell}\cdot\frac{\ell}{2}} - \cancel{\cos\frac{n\pi}{\ell}\ell}\right)$$

$$- \frac{4h}{\ell^2}\left\{\frac{\ell}{n\pi}\left(\cancel{\frac{\ell}{2}\cos\left(\frac{n\pi}{\ell}\frac{\ell}{2}\right)} - \cancel{\ell\cos\left(\frac{n\pi}{\ell}\ell\right)}\right)\right.$$

$$\left. + \frac{\ell^2}{n^2\pi^2}\left(\cancel{\sin\left(\frac{n\pi}{\ell}\ell\right)} - \sin\left(\frac{n\pi}{\ell}\frac{\ell}{2}\right)\right)\right\}$$

$$= \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$\Rightarrow \quad \beta_n = \frac{8h\ell}{n^3\pi^3 c}\sin\left(\frac{n\pi}{2}\right)$ as required