Handout

(2) (1) $\sin 2\theta$ = it's own Fourier series

(2) $$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

(3) $$\sin^3\theta = \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^3 = \frac{e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}}{-8i}$$

$$= -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta .$$

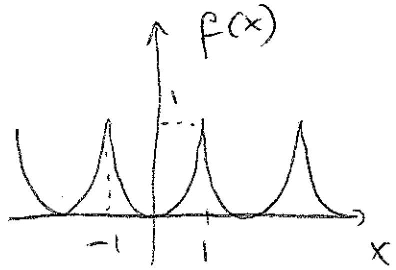
(5) $$\int_0^1 x^2 \cos(n\pi x)\,dx = \left[x^2 \cdot \frac{1}{n\pi}\sin(n\pi x)\right]_0^1 - \int_0^1 \frac{2x}{n\pi}\sin(n\pi x)\,dx$$

$$= 0 + \left[\frac{2x}{n^2\pi^2}\cos(n\pi x)\right]_0^1 - \int_0^1 \frac{2}{n^2\pi^2}\cos(n\pi x)\,dx$$

$$= \frac{2}{n^2\pi^2}(-1)^n - \left[\frac{2}{n^3\pi^3}\sin(n\pi x)\right]_0^1$$

$$= (-1)^n \cdot \frac{2}{n^2\pi^2}$$

(6)



$$f(x) = a_0 + \sum_1^{\infty}{}' a_n \cos(nx\pi)$$

by symmetry and $L=1$

$$a_0 = \frac{1}{2}\int_{-1}^1 f(x)\,dx = \int_0^1 x^2\,dx = \frac{1}{3}$$

$$a_n = \frac{1}{1}\int_{-1}^1 f(x)\,dx = 2\int_0^1 x^2\cos(n\pi x)\,dx$$

$$= \frac{4(-1)^n}{n^2\pi^2}$$

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty}{}' \frac{4(-1)^n}{n^2\pi^2}\cos(n\pi x) \Leftarrow$$

(7) $$f(x) = \sum_n{}' b_n \sin(n\pi x) \qquad g(x) = \sum_m{}' B_n \sin(n\pi x)$$

$$\int_{-1}^1 f(x)g(x)\,dx = \int_{-1}^1 \sum_n{}' \sum_m{}' b_n B_m \sin(n\pi x)\sin(m\pi x) = \sum_{n=1}^{\infty}{}' b_n B_n$$

since $\langle \sin(n\pi x)\sin(m\pi x)\rangle = \delta_{mn}$.

For $f = g = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$ then

$$\int_{-1}^{1} f(x)g(x)\,dx = \int_{-1}^{1} \left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x\right)^2 dx$$

$$= \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2$$

(2) let $f(x) = g(x) = x^2$ then

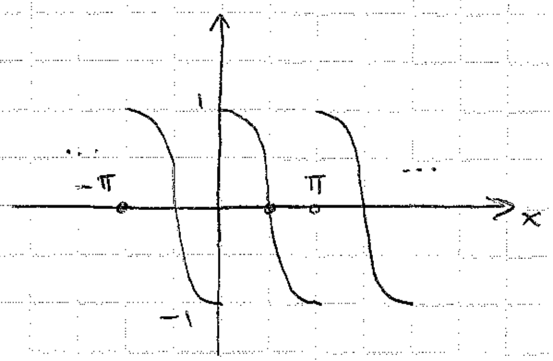
$$\int_{-1}^{1} x^4\,dx = \left(\frac{x^5}{5}\right)_{-1}^{1} = \frac{2}{5}$$

but is also equal to $= 2\left(\frac{1}{9}\right) + \sum\left(\frac{4(-1)^n}{n^2\pi^2}\right)^2$

so $$\frac{2}{5} - \frac{2}{9} = \sum_{n=1}^{\infty} \frac{16}{n^4\pi^4}$$

$$\Rightarrow \frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

(9) $f(x) = \cos x$ for $x \in [0,\pi]$
$= -\cos x$ for $x \in [-\pi, 0]$



$\Rightarrow f(x) = \sum b_n \sin(n x)$

($L = \pi$; odd function)

with $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx = \frac{2}{\pi}\int_0^{\pi} \cos x \sin(nx)$

$$= \frac{2}{\pi}\int_0^{\pi} \left[\sin((1+n)x) - \sin((1-n)x)\right]dx$$

$$= \frac{2}{\pi}\left\{\left[\frac{-1}{1+n}\cos(nx)\right]_0^1 - \left[\frac{-1}{1-n}\cos(nx)\right]_0^1\right\}$$

$$\begin{cases} = \dfrac{4n}{\pi(n^2-1)} & \text{if } n \text{ is even} \\ = 0 & \text{otherwise} \end{cases}$$

$\Rightarrow f(x) = \sum \frac{4n}{\pi(n^2-1)} \sin(nx)$. Note that $f(0) = 0$ in the series.