$\boxed{3.1}$ $u_{xx} - 6u_{xy} + 9u_{yy} = xy^2$

(a) $\delta(\mathcal{L}) = 0 \Rightarrow$ a parabolic system.

To find the relevant change of variables $(x, y) \to (\xi, \eta)$, we need to solve

$$\eta_x - 3\eta_y = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -3 \quad \Rightarrow \quad y = -3x + \eta$$

To find ξ, we can take any function of (x, y) that always intercepts lines of constant $\eta = y + 3x$ $\Rightarrow$ take $\xi = x$

Then $$\partial_x = \frac{\partial\eta}{\partial x}\frac{\partial}{\eta} + \frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi} = 3\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\xi}$$

$$\partial_y = \frac{\partial}{\partial\eta}$$

So note also that $u_{xx} - 6u_{xy} + 9u_{yy}$
$= (\partial_x - 3\partial_y)^2 u$

so

$$\partial_x - 3\partial_y = 3\partial_\eta + \partial_\xi - 3\partial_\eta = \partial_\xi$$

So the LHS of the operator is simply $\partial_{\xi\xi}$

and $$\begin{cases} \eta = y + 3x \\ \xi = x \end{cases} \Rightarrow \begin{cases} y = \eta - 3\xi \\ x = \xi \end{cases}$$

$\Rightarrow$ the canonical form of the equation is

$$u_{\xi\xi} = \xi(\eta - 3\xi)^2$$

This is not quite the required form:

To put it into the required form, simply set

$$t = \eta - 3\xi \quad \text{and} \quad \frac{\xi}{3} = s - t$$

So $\boxed{9u_{tt} = \frac{1}{3}(s-t)t^2}$ (works since $u_\xi = -3u_t \Rightarrow u_{\xi\xi} = 9u_{tt}$)

(b) To find the general solution, integrate wrt t:

$$u_{tt} = \frac{1}{27}(st^2 - t^3)$$

$$\Rightarrow u_t = \frac{1}{27}\left(s\frac{t^3}{3} - \frac{t^4}{4}\right) + h_1(s)$$

$$u = \frac{1}{27}\left(s\frac{t^4}{12} - \frac{t^5}{20}\right) + h_1(s)t + h_2(s)$$

$$\Rightarrow u(x,y) = \frac{1}{27}\left(\frac{(y+3x)y^4}{12} - \frac{y^5}{20}\right) + h_1(y+3x)y + h_2(y+3x)$$

Since $t = y$ and $s = y+3x$

(c) To fit to the bcs:

- $\sin x = h_2(\ 3x) \Rightarrow \quad h_2(x) = \sin\left(\frac{x}{3}\right)$
- $u_y(x,y) = \frac{1}{27}\left[\frac{y^3}{3}(y+3x) + \frac{y^4}{12} - \frac{y^4}{4}\right] + h_1(y+3x)$
  $\quad + y h_1'(y+3x) + h_2'(y+3x)$

so $\cos x = h_1(\ 3x) + \frac{1}{3}\cos\left(\frac{x}{3}\right) \Rightarrow h_1(x) = \frac{2}{3}\cos\left(\frac{x}{3}\right)$

So finally $u(x,y) = \frac{1}{27}\left[\frac{(y+3x)y^4}{12} - \frac{y^5}{20}\right] + \frac{2}{3}y\cos\left(\frac{y+3x}{3}\right) + \sin\left(\frac{y+3x}{3}\right)$

$\boxed{32}$ $u_{xx} + 6u_{xy} - 16u_{yy} = 0$

(a) $\delta(\mathcal{L}) = 9 + 16 = 25 > 0 \Rightarrow$ hyperbolic

(b) To find the canonical form, we need to solve the two characteristics equations:

$$\xi_x + (3+5)\,\xi_y = 0 \qquad ①$$

$$\eta_x + (3-5)\,\eta_y = 0 \qquad ②$$

$① \Rightarrow \frac{dy}{dx} = 8 \Rightarrow y = 8x + \xi \qquad \xi = y - 8x$

$② \Rightarrow \frac{dy}{dx} = -2 \Rightarrow y = -2x + \eta \qquad \eta = y + 2x$

So $\partial_x = -8\partial_\xi + 2\partial_\eta$

$\partial_y = \partial_\xi + \partial_\eta$

$\partial_{xx} = (-8\partial_\xi + 2\partial_\eta)(-8\partial_\xi + 2\partial_\eta)$

$= 64\,\partial_{\xi\xi} - 32\partial_{\xi\eta} + 4\partial_{\eta\eta}$

$\partial_{yy} = \partial_{\xi\xi} + 2\partial_{\xi\eta} + \partial_{\eta\eta}$

$\partial_{xy} = (-8\partial_\xi + 2\partial_\eta)(\partial_\xi + \partial_\eta)$

$= -8\partial_{\xi\xi} - 6\partial_{\xi\eta} + 2\partial_{\eta\eta}$

So $u_{xx} + 6u_{xy} - 16u_{yy} = \cancel{64u_{\xi\xi}} - 32u_{\eta\xi} + \cancel{4u_{\eta\eta}}$

$- \cancel{48u_{\xi\xi}} - 36u_{\eta\xi} + \cancel{12u_{\eta\eta}}$

$- \cancel{16u_{\xi\xi}} - 32u_{\eta\xi} - \cancel{16u_{\eta\eta}}$

$\Rightarrow 100\,u_{\eta\xi} = 0$

(c) So $u = F(\eta) + G(\xi) = F(y+2x) + G(y-8x)$

(d) To satisfy $u(-s, 2s) = s$

$u(s, 0) = \sin(2s)$

we require

$$s = F(2s - 2s) + G(2s + 8s)$$

$$\sin(2s) = F(0 + 2s) + G(0 - 8s)$$

the first equation implies

$$G(10s) = s + \text{constant} = s - F(0)$$

$$\Rightarrow \quad G(s) = \frac{s}{10} - F(0)$$

the second implies

$$\sin(2s) = F(2s) + G(-8s)$$

$$= F(2s) - \frac{8s}{10} - F(0) = F(2s) - \frac{4s}{5} - F(0)$$

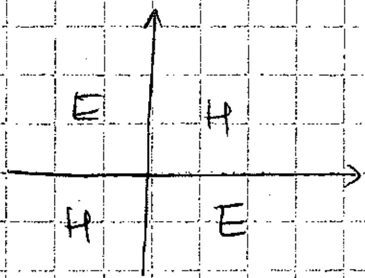
So $\quad F(s) = \sin(s) + \frac{2s}{5} + F(0)$

and finally

$$u(x,y) = \sin(y+2x) + \frac{2}{5}(y+2x) + \frac{(y-8x)}{10}$$

$\boxed{3.5}$ $$xu_{xx} - yu_{yy} + \frac{1}{2}(u_x - u_y) = 0$$

(a) $\delta(\mathcal{L}) = xy \Rightarrow$



$\delta(\mathcal{L}) > 0$ when $\begin{cases} x > 0 \text{ and } y > 0 \\ \text{or } x < 0 \text{ and } y < 0 \end{cases}$ <u>hyperbolic</u>

$\delta(\mathcal{L}) < 0$ when $\begin{cases} x > 0 \text{ and } y < 0 \\ \text{or } x < 0 \text{ and } y > 0 \end{cases}$ <u>elliptic</u>

(b) In the hyperbolic region, we solve for two families of characteristics:

$$x\xi_x + \sqrt{xy}\,\xi_y = 0 \qquad ①$$
$$x\eta_x - \sqrt{xy}\,\eta_y = 0 \qquad ②$$

① $\Rightarrow \dfrac{dy}{dx} = \dfrac{\sqrt{xy}}{x} = \sqrt{\dfrac{y}{x}}$ so $\dfrac{dy}{\sqrt{y}} = \dfrac{dx}{\sqrt{x}}$

$$\Rightarrow y^{1/2} = x^{1/2} + \xi$$

② $\Rightarrow \dfrac{dy}{dx} = -\sqrt{\dfrac{y}{x}}$ so $y^{1/2} = -x^{1/2} + \eta$

$$u_x = \frac{\partial\xi}{\partial x}u_\xi + \frac{\partial\eta}{\partial x}u_\eta = -\frac{1}{2}x^{-\frac{1}{2}}u_\xi + \frac{1}{2}x^{-1/2}u_\eta$$
$$u_y = \frac{\partial\xi}{\partial y}u_\xi + \frac{\partial\eta}{\partial y}u_\eta = \frac{1}{2}y^{-\frac{1}{2}}u_\xi + \frac{1}{2}y^{-\frac{1}{2}}u_\eta$$

$$u_{xx} = \frac{1}{4x}u_{\xi\xi} + 2\left(-\frac{1}{4x}\right)u_{\eta\xi} + \frac{1}{4x}u_{\eta\eta} + \frac{1}{4}x^{-\frac{3}{2}}u_\xi - \frac{1}{4}x^{-\frac{3}{2}}u_\eta$$
$$u_{yy} = \frac{1}{4y}u_{\xi\xi} + 2\left(\frac{1}{4y}\right)u_{\eta\xi} + \frac{1}{4y}u_{\eta\eta} - \frac{1}{4}y^{-\frac{3}{2}}u_\xi - \frac{1}{4}y^{-\frac{3}{2}}u_\eta$$

so $\quad x u_{xx} - y u_{yy} + \frac{1}{2}(u_x - u_y) = 0$

$$\Rightarrow \frac{1}{4}\left[\cancel{u_{\xi\xi}} - 2u_{\eta\xi} + \cancel{u_{\eta\eta}} + \cancel{x^{-1/2} u_\xi} - \cancel{x^{-1/2} u_\eta}\right]$$

$$- \frac{1}{4}\left[\cancel{u_{\xi\xi}} + 2u_{\eta\xi} + \cancel{u_{\eta\eta}} - \cancel{y^{-1/2} u_\xi} - \cancel{y^{-1/2} u_\eta}\right]$$

$$+ \frac{1}{2}\left[-\cancel{\frac{1}{2}x^{-1/2} u_\xi} + \cancel{\frac{1}{2}x^{-1/2} u_\eta} - \cancel{\frac{1}{2}y^{-\frac{1}{2}} u_\xi} - \cancel{\frac{1}{2}y^{-\frac{1}{2}} u_\eta}\right] = 0$$

$$\Rightarrow \boxed{- \quad u_{\eta\xi} = 0}$$

(c) In the elliptic domain, we must solve for the complex function ϕ the equation

$$x\phi_x \pm i\sqrt{xy}\,\phi_y = 0$$

(Here we take the $x>0$, $y<0$; the other is obtained in similar way)

$$\Rightarrow \frac{dy}{dx} = \pm i\sqrt{\frac{(-y)}{x}} \quad \text{so} \quad (-y)^{1/2} = \pm i x^{1/2} + \phi$$

Let's choose the $-$ solut° for simplicity.

$$\phi = (-y)^{1/2} + i x^{1/2} \Rightarrow \begin{cases} \xi = (-y)^{1/2} & y = -\xi^2 \\ \eta = x^{1/2} & x = \eta^2 \end{cases}$$

$$u_x = \frac{\partial\xi}{\partial x} u_\xi + \frac{\partial\eta}{\partial x} u_\eta = \frac{1}{2}x^{-\frac{1}{2}} u_\eta = \frac{1}{2\eta} u_\eta$$

$$u_y = \frac{\partial\xi}{\partial y} u_\xi + \frac{\partial\eta}{\partial y} u_\eta = -\frac{1}{2}(-y)^{-\frac{1}{2}} u_\xi = -\frac{1}{2\xi} u_\xi$$

so the equation becomes

$$\eta^2 \frac{1}{2\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{2\eta}\frac{\partial u}{\partial\eta}\right) + \xi^2 \frac{1}{2\xi}\frac{\partial}{\partial\xi}\left(\frac{1}{2\xi}\frac{\partial u}{\partial\xi}\right) + \frac{1}{2}\left(\frac{1}{2\eta}u_\eta + \frac{1}{2\xi}u_\xi\right) = 0$$

$$= \frac{1}{4}u_{\eta\eta} - \cancel{\frac{1}{4\eta}u_\eta} + \frac{1}{4}u_{\xi\xi} - \cancel{\frac{1}{4\xi}u_\xi} + \cancel{\frac{1}{4\eta}u_\eta} + \cancel{\frac{1}{4\xi}u_\xi} = 0$$

$$\Rightarrow \quad \frac{1}{4}(u_{\eta\eta} + u_{\xi\xi}) = 0$$

$\boxed{3.10}$ $u_{xx} - 2u_{xy} + 4e^{y} = 0$

$\delta(\mathcal{L}) = 1 \Rightarrow$ hyperbolic

$$\frac{dy}{dx} = \frac{-1 \pm \sqrt{1}}{1} = \begin{cases} -2 \\ 0 \end{cases}$$

$$\Rightarrow \begin{cases} y = -2x + \xi \\ y = \eta \end{cases} \Rightarrow \begin{cases} \xi = y + 2x \\ \eta = y \end{cases}$$

$$\begin{cases} \xi_x = 2 & \xi_y = 1 \\ \eta_x = 0 & \eta_y = 1 \end{cases}$$

$$u_{xx} = 4u_{\xi\xi} \qquad \Rightarrow \quad 4u_{\xi\xi} - 2(2u_{\xi\xi} + 2u_{\xi\eta})$$
$$u_{xy} = 2u_{\xi\xi} + 2u_{\xi\eta} \qquad + 4e^{y} = 0$$

$$\Rightarrow -4u_{\xi\eta} + 4e^{y} = 0$$

But $y = \eta$ so $\boxed{u_{\xi\eta} = e^{\eta}}$

Let's integrate w.r.t η: $u_{\xi} = e^{\eta} + F(\xi)$

———————— ξ : $u = \xi e^{\eta} + F(\xi) + G(\eta)$

Initial conditions:

$u(0,y) = f(y)$

$u_x(0,y) = g(y)$

$$\Rightarrow \boxed{u = F(y+2x) + G(y) + (y+2x)e^{y}}$$

$$\Rightarrow \begin{cases} f(y) = F(y) + G(y) + e^{y}y \\ g(y) = 2F'(y) + 2e^{y} \rightarrow F(y) = \frac{1}{2}\int_0^y g(y')\,dy' - e^{y} \end{cases}$$

$$\rightarrow G(y) = f(y) + (1-y)e^{y} - \frac{1}{2}\int_0^y g(y')\,dy'$$

So $u(x,y) = \frac{1}{2}\int^{y+2x} g(y')\,dy' + f(y) - e^{y+2x} + (1+2x)e^{y}$

$\boxed{3.12}$ $u_{xx} + y u_{yy} = 0$

(a) $\delta(\mathcal{L}) = -y$ so $\begin{cases} \text{hyperbolic when } y<0 \\ \text{elliptic when } y>0 \end{cases}$

Hyperbolic domain $(y<0)$

We solve $\begin{cases} \xi_x + \sqrt{-y}\, \xi_y = 0 & (1) \\ \eta_x - \sqrt{-y}\, \eta_y = 0 & (2) \end{cases}$

$$(1) \Rightarrow \frac{dy}{dx} = \sqrt{-y} \Rightarrow \frac{dy}{\sqrt{-y}} = dx$$
$$\Rightarrow -2\sqrt{-y} = x + \xi$$

$$(2) \Rightarrow \frac{dy}{dx} = -\sqrt{-y} \Rightarrow \frac{dy}{\sqrt{-y}} = -dx$$
$$\Rightarrow -2\sqrt{-y} = -x + \eta$$

$$u_x = \frac{\partial \xi}{\partial x} u_\xi + \frac{\partial \eta}{\partial x} u_\eta = -u_\xi + u_\eta$$
$$u_y = \frac{\partial \xi}{\partial y} u_\xi + \frac{\partial \eta}{\partial y} u_\eta = \frac{1}{\sqrt{-y}} u_\xi + \frac{1}{\sqrt{-y}} u_\eta$$

$$u_{xx} = u_{\xi\xi} - 2u_{\eta\xi} + u_{\eta\eta}$$
$$u_{yy} = -\frac{1}{y} u_{\xi\xi} - \frac{2}{y} u_{\eta\xi} - \frac{1}{y} u_{\eta\eta} + \frac{1}{2}(-y)^{-\frac{3}{2}} u_\xi + \frac{1}{2}(-y)^{-\frac{3}{2}} u_\eta$$

note that $-4\sqrt{-y} = \xi + \eta$ so

$$u_{xx} + y u_{yy} = -2u_{\eta\xi} - 2u_{\eta\xi} + \frac{1}{2} y (-y)^{-\frac{3}{2}} u_\xi + \frac{1}{2} y (-y)^{-\frac{3}{2}} u_\eta$$
$$= -4u_{\eta\xi} - \frac{1}{2\sqrt{-y}} (u_\xi + u_\eta) = \underbrace{-4u_{\eta\xi} + \frac{2}{\xi+\eta} (u_\xi + u_\eta)}_{=0}$$

Elliptic domain $(y>0)$

We solve $\phi_x \pm i\sqrt{y}\,\phi_y = 0$

$$\Rightarrow \frac{dy}{dx} = \pm i\sqrt{y}$$

$$\Rightarrow \frac{dy}{\sqrt{y}} = \pm i\,dx$$

$$\Rightarrow 2\sqrt{y} = \pm ix + \phi$$

choose − solut°:

$$\phi = 2\sqrt{y} + ix \quad \Rightarrow \begin{cases} \xi = 2\sqrt{y} \\ \eta = x \end{cases}$$

So $u_x = \frac{\partial \xi}{\partial x} u_\xi + \frac{\partial \eta}{\partial x} u_\eta = u_\eta$

$u_y = \frac{\partial \xi}{\partial y} u_\xi + \frac{\partial \eta}{\partial y} u_\eta = \frac{1}{\sqrt{y}} u_\xi$

$$u_{xx} = u_{\eta\eta}$$

$$u_{yy} = \frac{1}{y} u_{\xi\xi} - \frac{1}{2y^{3/2}} u_\xi$$

So

$$u_{xx} + y u_{yy} = u_{\eta\eta} + u_{\xi\xi} - \frac{1}{2\sqrt{y}} u_\xi$$

$$= u_{\eta\eta} + u_{\xi\xi} - \frac{1}{\xi} u_\xi = 0$$

Problem 7.14 , Handout

let $X(t)$ be population size at time t

Define $P(X(t)=n) = P_n(t)$

Then let :

Probability that during $[t, t+dt]$, with $X(t)=n$, there is one birth, is $\lambda_n(t)dt$

Probability that during $[t, t+dt]$, with $X(t)=n$, there is one death, is $\mu_n(t)dt$

$$P_0(t+dt) = (\text{Probability that nothing happens}) \cdot P_0(t) + (\text{Probability of 1 death}) \cdot P_1(t)$$

$$\Rightarrow P_0(t+dt) = P_0(t)\,(1-\lambda_0(t)dt - \mu_0(t)dt) + \mu_1(t)P_1(t)dt$$

$$\Rightarrow \frac{P_0(t+dt)-P_0(t)}{dt} = -(\lambda_0(t)+\mu_0(t))P_0(t) + \mu_1(t)P_1(t)$$

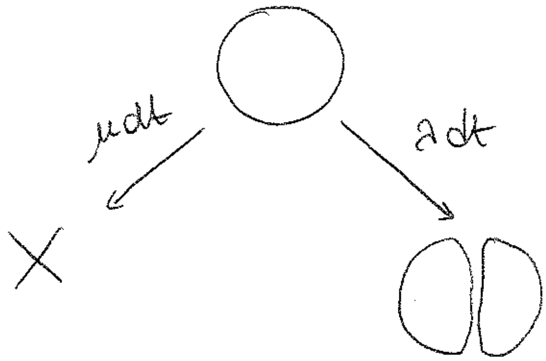
$$P_n(t+dt) = (\text{Probability that nothing happens}) \cdot P_n(t) + (\text{Probability of 1 birth}) \cdot P_{n-1}(t) + (\text{Probability of 1 death}) \cdot P_{n+1}(t)$$

$$= (1-\lambda_n(t)dt - \mu_n(t)dt)\,P_n(t) + \lambda_{n-1}(t)dt\,P_{n-1}(t) + \mu_{n+1}(t)\,dt\,P_{n+1}(t)$$

$$\Rightarrow \frac{P_n(t+dt)-P_n(t)}{dt} = -(\lambda_n(t)+\mu_n(t))P_n(t) + \lambda_{n-1}(t)P_{n-1}(t) + \mu_{n+1}(t)P_{n+1}(t)$$

Problem 7.16 Handout

(a)



for 1 bactera, probability of dying during $[t, t+dt]$ is μdt and probability of splitting in two (mitosis) is λdt

$\rightarrow$ probability of 1 birth is $n\lambda dt$ (since there are n bactera)

$\rightarrow$ probability of 1 death is $n\mu dt$ (same)

$$\Rightarrow \quad \lambda_n(t) = n\lambda$$
$$\mu_n(t) = n\mu \quad \text{in notation of 7.14.}$$

(b) $$G(t,s) = \sum_{n=0}^{\infty} P_n(t) s^n$$

$$\Rightarrow \frac{\partial G}{\partial t} = \sum_{n=0}^{\infty} P_n'(t) s^n = P_0'(t) + \sum_{n=1}^{\infty} P_n'(t) s^n$$

$$= -(\lambda_0(t) + \mu_0(t)) P_0(t) + \mu_1(t) P_1(t)$$
$$+ \sum_{n=1}^{\infty} \Big(-(\lambda_n(t) + \mu_n(t)) P_n(t) + \lambda_{n-1}(t) P_{n-1}(t) + \mu_{n+1}(t) P_{n+1}(t) \Big) s^n$$

since $\lambda_0(t) = 0$, $\mu_0(t) = 0$ and as above $\rightarrow$

$$= \mu P_1(t) + \sum_{n=1}^{\infty} \Big[-n(\lambda+\mu) P_n(t) + \lambda(n-1) P_{n-1}(t) + \mu(n+1) P_{n+1}(t) \Big] s^n$$

$$= \sum_{n=0}^{\infty} \Big[-n(\lambda+\mu) P_n(t) + \mu(n+1) P_{n+1}(t) \Big] s^n + \lambda n P_n(t) s^{n+1}$$

$$= -s(\lambda+\mu)\frac{\partial G}{\partial s} + \mu \frac{\partial G}{\partial s} + \lambda s^2 \frac{\partial G}{\partial s}$$

$$\Rightarrow \quad \frac{\partial G}{\partial t} + (1-s)(\lambda s - \mu)\frac{\partial G}{\partial s} = 0$$, as required

If At $t=0$ there are n_0 bactera then

$$\begin{cases} P_{n_0}(t=0)=1 \\ P_n(t=0)=0 \quad \forall n \neq 0 \end{cases} \Rightarrow G(0,s)=s^{n_0}$$

(c) We solve the PDE

$$\begin{cases} \frac{\partial G}{\partial t} + (1-x)(\lambda x-\mu)\frac{\partial G}{\partial x} = 0 \\ G(t=0,x) = x^{n_0} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{dt}{d\tau} = 1 \quad \Rightarrow \quad t = \tau \\ \frac{dG}{d\tau} = 0 \quad \Rightarrow \quad G = G_0(s) = s^{n_0} \\ \frac{dx}{d\tau} = (1-x)(\lambda x-\mu) \quad \Rightarrow \quad \frac{dx}{(1-x)(\lambda x-\mu)} = d\tau \end{cases}$$

$$\frac{A}{1-x} + \frac{B}{\lambda x-\mu} = \frac{1}{(1-x)(\lambda x-\mu)}$$

$$\Rightarrow A(\lambda x-\mu) + B(1-x) = 1$$

$$\Rightarrow A\lambda - B = 0 \quad \text{and} \quad -A\mu + B = 1$$

$$\Rightarrow B = A\lambda \qquad A(\lambda-\mu) = 1 \Rightarrow A = \frac{1}{\lambda-\mu}$$

$$\Rightarrow B = \frac{\lambda}{\lambda-\mu}$$

So $$\frac{1}{\lambda-\mu}\frac{dx}{1-x} + \frac{\lambda}{\lambda-\mu}\frac{dx}{\lambda x-\mu} = d\tau$$

$$\Rightarrow -\ln(1-x) + \ln(\lambda x-\mu) = (\lambda-\mu)\tau + k$$

$$\Rightarrow \frac{\lambda x-\mu}{1-x} = e^{(\lambda-\mu)\tau} \cdot \tilde{k}$$

at $\tau = 0$ $x = s \Rightarrow$ $$\frac{\lambda x-\mu}{1-x} = \frac{\lambda s-\mu}{1-s} e^{(\lambda-\mu)\tau}$$

So
$$(1-s)\left(\frac{\lambda x-\mu}{1-x}\right) = (\lambda s-\mu)e^{(\lambda-\mu)\tau}$$

$$\Rightarrow \quad s\left[\frac{\lambda x-\mu}{1-x} + \lambda e^{(\lambda-\mu)\tau}\right] = \left[\mu e^{(\lambda-\mu)\tau} + \frac{\lambda x-\mu}{1-x}\right]$$

$$\Rightarrow \quad s = \frac{(\lambda x-\mu) + (1-x)\mu e^{(\lambda-\mu)\tau}}{(\lambda x-\mu) + \lambda(1-x)e^{(\lambda-\mu)\tau}}$$

So
$$G = s^{n_0} = \left[\frac{(\lambda x-\mu) + (1-x)\mu e^{(\lambda-\mu)t}}{(\lambda x-\mu) + (1-x)\lambda e^{(\lambda-\mu)t}}\right]^{n_0}$$

Finally, recall that we "called" s, x

$$\Rightarrow \quad G(t,s) = \left[\frac{(\lambda\ -\mu)+(1-\ )\mu e^{(\lambda-\mu)t}}{(\lambda\ -\mu)+(1-\ )\lambda e^{(\lambda-\mu)t}}\right]^{n_0}.$$

Expectation value: $\quad E(t) = \left.\frac{\partial G}{\partial s}\right|_{s=1}$

so calculate $\frac{\partial G}{\partial s}$ using logarithmic derivatives

$$\ln G = n_0 \ln\left[(\lambda s-\mu) + (1-s)\mu e^{(\lambda-\mu)t}\right] - n_0 \ln\left[(\lambda s-\mu) + (1-s)\lambda e^{(\lambda-\mu)t}\right]$$

$$\frac{G'}{G} = n_0\frac{\lambda - \mu e^{(\lambda-\mu)t}}{(\lambda s-\mu)+(1-s)\mu e^{(\lambda-\mu)t}} - n_0\frac{\lambda - \lambda e^{(\lambda-\mu)t}}{(\lambda s-\mu)+(1-s)\lambda e^{(\lambda-\mu)t}}$$

$G(s=1) = 1$ so

$$\left.\frac{\partial G}{\partial s}\right|_{s=1} = \frac{n_0\left(\lambda - \mu e^{(\lambda-\mu)t}\right)}{\lambda-\mu} - \frac{n_0\left(\lambda - \lambda e^{(\lambda-\mu)t}\right)}{\lambda-\mu}$$

$$= n_0 e^{(\lambda-\mu)t} \quad \text{as required}$$