Homework 3

Problem 1 moved to HW4

Problem 2

$$\begin{cases} x u u_x + y u u_y = u^2 - 1 \\ u(x, x^2) = x^3 \quad \text{for } x > 0 \end{cases}$$

let $v = u^2 - 1$ then

$$\begin{matrix} v_x = u u_x \\ v_y = u u_y \end{matrix} \Rightarrow \frac{x}{2} v_x + \frac{y}{2} v_y = v$$

↑ a linear equation!

The initial conditions become

$$v(x, x^2) = x^6 - 1$$

$$\Rightarrow \begin{cases} x_0(s) = s \\ y_0(s) = s^2 \\ v_0(s) = s^6 - 1 \end{cases}$$

Transersality condition:

$$\begin{vmatrix} 1 & \frac{s}{2} \\ 2s & \frac{s^2}{2} \end{vmatrix} = \frac{s^2}{2} - s^2 = -\frac{s^2}{2}$$

↳ problem at $x=0$ or $y=0$.

Characteristic equations

$$\begin{cases} \frac{dx}{d\tau} = \frac{x}{2} \Rightarrow x = s e^{\tau/2} \\ \frac{dy}{d\tau} = y/2 \Rightarrow y = s^2 e^{\tau/2} \\ \frac{dv}{d\tau} = v \Rightarrow v = (s^6 - 1) e^{\tau} \end{cases}$$

So $v(s, \tau) = (s^6 - 1) e^{\tau}$

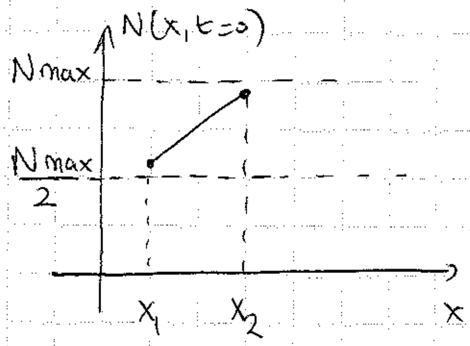
$s = \frac{y}{x}$ $\quad \tau = 2 \ln\left(\frac{x}{s}\right)$ $\quad e^{\tau} = \frac{x^2}{s^2} = \frac{x^4}{y^2}$

$$\Rightarrow v(x, y) = \left(\left(\frac{y}{x}\right)^6 - 1 \right) \frac{x^4}{y^2}$$

$$\Rightarrow u(x, y) = \sqrt{v + 1} = \sqrt{\frac{x^4}{y^2}\left(\frac{y^6}{x^6} - 1\right) + 1} = \sqrt{\frac{y^4}{x^2} - \frac{x^4}{y^2} + 1}$$

Problem 3

(a)



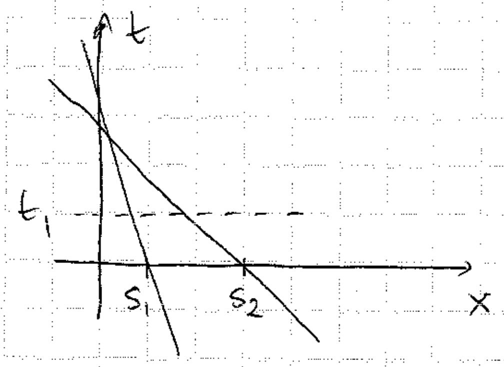
$N(x_1, t=0) < N(x_2, t=0)$

$\Rightarrow \phi(s_1) < \phi(s_2)$

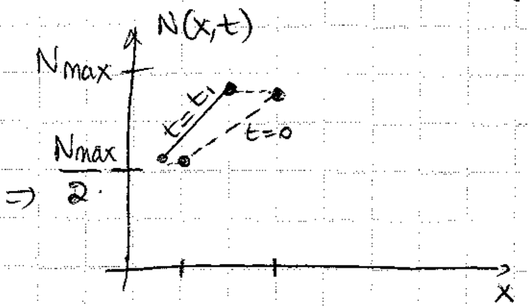
$\Rightarrow F'(\phi(s_1)) > F'(\phi(s_2))$

$$\Rightarrow \frac{1}{F'(\phi(s_1))} < \frac{1}{F'(\phi(s_2))}$$

(but both now negative)

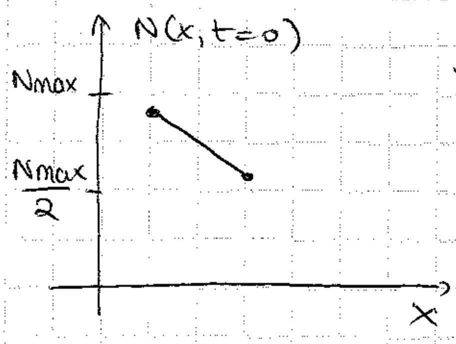


$\Rightarrow$

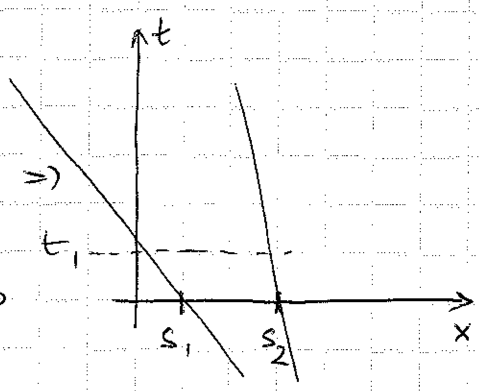


→ the front steepens & moves backward

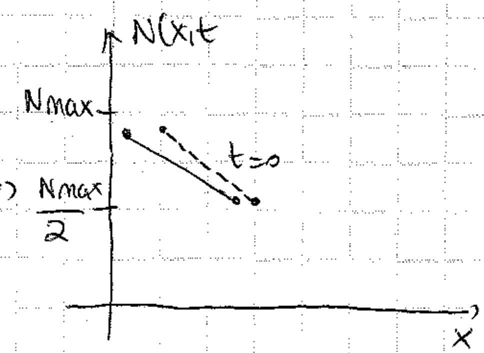
(b) By contrast



$\Rightarrow$



$\Rightarrow$



→ The front moves backward but becomes shallower.

Problem 7.1 Handout

Derive $P_0'(t) = -\lambda P_0(t) + \mu P_1(t)$

Assumptions:

- if a line is occupied, probability that conversation ends is μh
- probability that a conversation starts is λh

$\rightarrow$ $P_0(t+dt) =$ (Probability that 1 line was occupied at t) $\cdot$ (Probability it ends)

$+$ (Probability that 0 lines were occupied at t) $\cdot$ (Probability no new ones start)

$$= P_1(t)\cdot \mu dt + P_0(t)\cdot(1-\lambda dt)$$

$$\Rightarrow P_0(t+dt) - P_0(t) = [\mu P_1(t) - \lambda P_0(t)]dt$$

$$\frac{dP_0}{dt} = -\lambda P_0 + \mu P_1 .$$

Problem 7.2

Derive $\frac{\partial G}{\partial t} + \mu(s-1)\frac{\partial G}{\partial s} = \lambda(s-1)G$

from $G(t,s) = \sum_{n=0}^{\infty} P_n(t)s^n$

$$\Rightarrow \frac{\partial G}{\partial t} = \sum_{n=0}^{\infty} P_n'(t)s^n$$

$$\frac{\partial G}{\partial s} = \sum_{n=1}^{\infty} nP_n(t)s^{n-1} = \sum_{n=0}^{\infty}(n+1)P_{n+1}(t)s^n$$

$$\frac{\partial G}{\partial t} = \sum_{n=1}^{\infty}\left[-(\lambda+n\mu)P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t)\right]s^n$$

$$+ (-\lambda P_0 + \mu P_1)$$

$\Rightarrow$

combine with $-\lambda P_0$ term

$$\frac{\partial G}{\partial t} = \sum_{n=0}^{\infty} -\lambda P_n(t)s^n - n\mu P_n(t)s^n$$

$$+ \lambda \sum_{n=0}^{\infty} P_n(t)s^{n+1} + \mu \sum_{n=0}^{\infty} (n+1)P_{n+1}(t)s^n$$

change index in $\lambda P_{n-1}s^n$ term

combine with μP_1 term

$$= -\lambda G - \mu s\frac{\partial G}{\partial s} + \lambda s G + \mu\frac{\partial G}{\partial s}$$

$$\boxed{\frac{\partial G}{\partial t} = \lambda(s-1)G + \mu(1-s)\frac{\partial G}{\partial s}}$$

Problem 7.3

Let's solve $\frac{\partial G}{\partial t} + \mu(s-1)\frac{\partial G}{\partial s} = \lambda(s-1)G$

with initial condition $G(0,s) = g(s)$

Initial condition curve: $\Gamma: \begin{cases} t_0(\xi) = 0 \\ s_0(\xi) = \xi \\ G_0(\xi) = g(\xi) \end{cases}$

Characteristic equations:

$$\begin{cases} \frac{dt}{d\tau} = 1 & \rightarrow \quad t = \tau + t_0(\xi) = \tau \\ \frac{ds}{d\tau} = \mu(s-1) & \rightarrow \quad s = 1 + (\xi-1)e^{\mu\tau} \\ \frac{dG}{d\tau} = \lambda(s-1)G & \end{cases}$$

since

$$\frac{ds}{s-1} = \mu\, d\tau \quad \rightarrow \quad \ln(s-1) = \mu\tau + K$$

$$s-1 = \tilde{K}e^{\mu\tau}$$

at $\tau = 0$ $s = \xi$ so

$$\tilde{K} = \xi - 1$$

Now $\frac{dG}{d\tau} = \lambda(\xi-1)e^{\mu\tau} G$

$$\Rightarrow \quad \frac{dG}{G} = \lambda(\xi-1)e^{\mu\tau}\, d\tau$$

$$\ln G = \frac{\lambda}{\mu}(\xi-1)e^{\mu\tau} + k'$$

$$G = \tilde{k}' e^{\frac{\lambda}{\mu}(\xi-1)e^{\mu\tau}}$$

at $\tau = 0$ $G = g(\xi) \Rightarrow g(\xi) = \tilde{k}' e^{\frac{\lambda}{\mu}(\xi-1)}$

$$\rightarrow \tilde{k}' = g(\xi) e^{-\frac{\lambda}{\mu}(\xi-1)}$$

So $G(\xi,\tau) = g(\xi) e^{\frac{\lambda}{\mu}(\xi-1)(e^{\mu\tau}-1)}$

Inverting for (t,s): $\tau = t$

$$\xi = 1 + (s-1)e^{-\mu t}$$

so $G(t,s) = g\left(1+(s-1)e^{-\mu t}\right) e^{\frac{\lambda}{\mu}(s-1)e^{-\mu t}(e^{\mu t}-1)}$

$$= g\left(1+(s-1)e^{-\mu t}\right) e^{\frac{\lambda}{\mu}(s-1)(1-e^{-\mu t})}$$

<u>Problem 7.4</u> Suppose $g(s) = s$ then

$$G(t,s) = \left[1+(s-1)e^{-\mu t}\right] e^{\frac{\lambda}{\mu}(s-1)(1-e^{-\mu t})}$$

$$= (a+bs)e^{c+ds}$$

where $a = 1-e^{-\mu t}$

$b = e^{-\mu t}$

$c = -\frac{\lambda}{\mu}(1-e^{-\mu t})$

$d = \frac{\lambda}{\mu}(1-e^{-\mu t})$

Near $s = 0$

$$G(t,s) = (a+bs)e^{c} \cdot \left(1 + ds + \frac{d^2 s^2}{2} + \cdots \frac{d^n s^n}{n!}\right)$$

$$= e^{c}\left[a + s(b+ad) + s^2\left(\frac{ad^2}{2} + bd\right) + \cdots s^n\left(\frac{ad^n}{n!} + \frac{bd^{n-1}}{(n-1)!}\right)\cdots\right]$$

So
$$P_0(t) = ae^{c} = (1-e^{-\mu t})e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})}$$

$$P_1(t) = (b+ad)e^{c} = \left(e^{-\mu t} + \frac{\lambda}{\mu}(1-e^{-\mu t})^2\right)e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})}$$

$$\vdots$$

$$P_n(t) = \frac{d^{n-1}}{(n-1)!}\left(b + \frac{ad}{n}\right)e^{c}$$

$$= \frac{1}{(n-1)!}\left(\frac{\lambda}{\mu}\right)^{n-1}(1-e^{-\mu t})^{n-1} \cdot \left[e^{-\mu t} + \frac{\frac{\lambda}{\mu}(1-e^{-\mu t})^2}{n}\right]e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})}$$

$$= \frac{1}{\mu n!}\left(\frac{\lambda}{\mu}\right)^{n-1}(1-e^{-\mu t})^{n-1} \cdot \left[n\mu e^{-\mu t} + \lambda(1-e^{-\mu t})^2\right]e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})}$$

Problem 7.7

As $t \to +\infty$

$$P_n(t) \to \frac{1}{\mu n!}\left(\frac{\lambda}{\mu}\right)^{n-1}(\lambda)e^{-\frac{\lambda}{\mu}} = \frac{\left(\frac{\lambda}{\mu}\right)^n e^{-\frac{\lambda}{\mu}}}{n!}$$

a Poisson process with parameter $\frac{\lambda}{\mu}$

Problem 7.5

(a) Suppose at time $t=0$ 2 lines are in use:

$$P_2(0) = 1 \quad \text{and} \quad P_n(0) = 0 \quad \forall n \neq 2.$$

then $g(s) = s^2$

So this time

$$G(t,s) = \left(1 + e^{-\mu t}(s-1)\right)^2 e^{\frac{\lambda}{\mu}(s-1)(1-e^{-\mu t})}$$

$$= (a+bs)^2 e^{c+ds}$$ with the same a, b, c, d.

→ near $s=0$

$$G(t,s) = e^c(a^2 + 2abs + b^2s^2)\left(1 + ds + \frac{d^2s^2}{2} + \dots \frac{d^ns^n}{n!} \dots\right)$$

$$\to P_n(t) = e^c\left[a^2\frac{d^n}{n!} + 2ab\frac{d^{n-1}}{(n-1)!} + b^2\frac{d^{n-2}}{(n-2)!}\right] \quad \text{for } n \geq 2$$

$$= e^c\frac{d^{n-2}}{(n-2)!}\left\{\frac{a^2d^2}{n(n-1)} + \frac{2abd}{(n-1)} + b^2\right\}$$

$$= \frac{1}{(n-2)!} e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})} \left(\frac{\lambda}{\mu}\right)^{n-2}(1-e^{-\mu t})^{n-2}$$

$$\cdot \left\{\frac{(1-e^{-\mu t})^4}{n(n-1)}\left(\frac{\lambda}{\mu}\right)^2 + \frac{2e^{-\mu t}\left(\frac{\lambda}{\mu}\right)(1-e^{-\mu t})^2}{n-1} + e^{-2\mu t}\right\}$$

as $t \to +\infty$

$$P_n(t) = \frac{1}{(n-2)!} e^{-\frac{\lambda}{\mu}} \left(\frac{\lambda}{\mu}\right)^{n-2}\left\{\frac{1}{n(n-1)}\left(\frac{\lambda}{\mu}\right)^2\right\}$$

$$= \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n e^{-\frac{\lambda}{\mu}}$$ → same final state.

(b) at $t=0$ $P_m(0)=1$, $P_n(0)=0$ $\forall n \neq m$.

$$\longrightarrow \quad g(s) = s^m \text{ and}$$

$$G(t,s) = \left(1 - e^{-\mu t}(s-1)\right)^m e^{\frac{\lambda}{\mu}(s-1)(1-e^{-\mu t})}$$

$$= (a+bs)^m e^{c+ds}$$

So $$G(t,s) = \left(\sum_{p=0}^{m} \binom{p}{m} a^p b^{m-p} s^{m-p}\right) e^{c+ds}$$

By analogy with previous question, and noting that $b \to 0$ as $t \to +\infty$,

$$\lim_{t\to+\infty} P_n(t) = a^m e^c \cdot \frac{d^n}{n!}.$$

$$= e^{-\frac{\lambda}{\mu}} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}$$ = another case of the same Poisson distribution

$\longrightarrow$ all initial conditions leads to the same Poisson distribution as $t \to +\infty$.

<u>Problem 7-6</u> $E(t) = \sum_{n=0}^{\infty} n P_n(t)$ = expectation value of # lines in use at time t

$$E(t) = \sum_{n=1}^{\infty} n P_n(t) (1)^n = \left.\frac{\partial G}{\partial s}\right|_{s=1}$$

Where $G(t,s) = \left(1+e^{-\mu t}(s-1)\right) e^{\frac{\lambda}{\mu}(s-1)(1-e^{-\mu t})}$

then $$\frac{\partial G}{\partial s} = e^{-\mu t} e^{\frac{\lambda}{\mu}(s-1)(1-e^{-\mu t})} + \frac{\lambda}{\mu}(1-e^{-\mu t})\left(1-e^{-\mu t}(s-1)\right) e^{\frac{\lambda}{\mu}(s-1)(1-e^{-\mu t})}$$

$$\Rightarrow \left.\frac{\partial G}{\partial s}\right|_{s=1} = e^{-\mu t} + \frac{\lambda}{\mu}\left(1-e^{-\mu t}\right)$$

$\longrightarrow$ as $t \to +\infty$ $E(t) \to \frac{\lambda}{\mu}$