EXERCISES p 21

1.1 Find a solution of the kind $u = f(ax+by)$

- $u_x + 3u_y = 0$ $\qquad \frac{\partial u}{\partial x} = af' \qquad \frac{\partial u}{\partial y} = bf'$

$$\Rightarrow af' + 3bf' = 0 \Rightarrow \boxed{a + 3b = 0} \rightarrow \text{for example, take } \begin{cases} b = 1 \\ a = -3 \end{cases}$$

- $3u_x - 7u_y = 0$

$$\Rightarrow 3af' - 7bf' = 0 \Rightarrow \boxed{3a - 7b = 0} \Rightarrow \text{for example take } \begin{cases} a = +7 \\ b = 3 \end{cases}$$

⋮

1.2 Find a solution of the kind $u = e^{\alpha x + \beta y}$

$$\frac{\partial u}{\partial x} = \alpha u \qquad \frac{\partial u}{\partial y} = \beta u$$

- $u_x + 3u_y + u = 0$

$$\Rightarrow \alpha u + 3\beta u + u = 0 \Rightarrow \boxed{\alpha + 3\beta + 1 = 0} \rightarrow \text{for example take } \begin{cases} \beta = 0 \\ \alpha = -1 \end{cases}$$

- $u_{xx} + u_{yy} = 5e^{x-2y}$

$$\alpha^2 u + \beta^2 u = 5e^{x-2y}$$

$$\Leftrightarrow (\alpha^2 + \beta^2) e^{\alpha x + \beta y} = 5e^{x-2y} \Rightarrow \begin{cases} \alpha^2 + \beta^2 = 5 \\ \alpha = 1 \\ \beta = 2 \end{cases} \text{ works.}$$

Note if we had had

$$\alpha^2 u + \beta^2 u = 4e^{x-2y}$$

$$\Leftrightarrow (\alpha^2 + \beta^2) e^{\alpha x + \beta y} = 4e^{x-2y} \quad \text{then write} \quad \begin{array}{l} \alpha = k\tilde{\alpha} \\ \beta = k\tilde{\beta} \end{array}$$

$$\Rightarrow k^2 e^k (\tilde{\alpha}^2 + \tilde{\beta}^2) e^{\tilde{\alpha} x + \tilde{\beta} y} = 4e^{x-2y}$$

and solve
$$\begin{cases} \tilde{\alpha} = 1 \\ \tilde{\beta} = -2 \\ 5k^2 e^k = 4 \quad \Rightarrow \quad k^2 e^k = \frac{4}{5} \quad \text{to find } k. \end{cases}$$

- $u_{xxxx} + u_{yyyy} + 2u_{xxyy} = 0$

$$\Rightarrow \quad \alpha^4 u + \beta^4 u + 2\alpha^2\beta^2 u = 0 \quad \Rightarrow \quad \alpha^4 + \beta^4 + 2\alpha^2\beta^2 = 0$$
$$\Rightarrow (\alpha^2 + \beta^2)^2 = 0$$
$$\Rightarrow \alpha = 0 \text{ and } \beta = 0$$

$\boxed{1.3}$ (a) Show that there exists a unique solution to
$$\begin{cases} u_x = 3x^2 y + y \\ u_y = x^3 + x \end{cases} \qquad u(0,0) = 0$$

$$u_y = x^3 + x \quad \Rightarrow \quad u = (x^3 + x)y + F_1(x)$$ ← arbitrary function of x only

$$u_x = 3x^2 y + y \quad \Rightarrow \quad u = x^3 y + xy + F_2(y)$$ ← arbitrary function of y only.

Identifying the two expressions, we see that
$$u = (x^3 + x)y + K$$
↖ constant

Matching to initial conditions $u(0,0) = 0 \Rightarrow K = 0$ which is the unique solution.

(b)
$$\begin{cases} u_x = 2.999999 x^2 y + y \\ u_y = x^3 + x \end{cases}$$

Similarly $\Rightarrow$
$$\begin{cases} u = \left(\frac{2.999999}{3} x^3 + x\right) y + G_1(x) \\ u = (x^3 + x) y + G_2(y) \end{cases}$$

There is no way of choosing G_1 an G_2 to identify the two expressions to each other $\rightarrow$ no solution to the system.

$\boxed{1.5}$ $\quad u_t = p(u)u_x \qquad t>0$

- Prove that $u = f(x + p(u)t)$ satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left[f\left(x + p(u(x,t))t\right)\right] = f'(\cdot)\left[p(u) + \frac{\partial p}{\partial u}\frac{\partial u}{\partial t}t\right]$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = \frac{p(u)\,f'}{1 - t\frac{\partial p}{\partial u}f'} \qquad \text{Similarly, } \frac{\partial u}{\partial x} = \frac{f'}{1 - t\frac{\partial p}{\partial u}f'}$$

so indeed $\frac{\partial u}{\partial t} = p(u)\frac{\partial u}{\partial x}$

- Use this to solve . $u_t = ku_x \Rightarrow p(u) = k$ so

$$u = f(x + kt)$$

. $u_t = uu_x \Rightarrow p(u) = u$ so

$$u = f(x + ut)$$

. $u_t = u\sin u\, u_x \Rightarrow p(u) = u\sin u$

$$u = f(x + u\sin u\, t).$$

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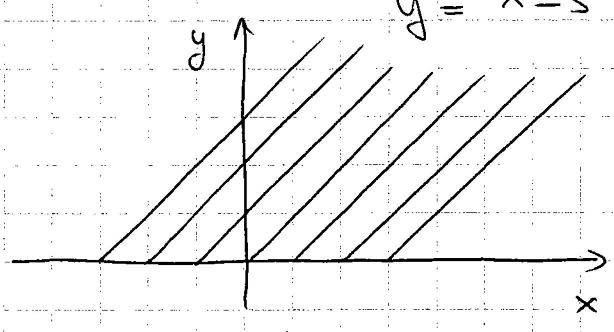
2.1 $u_x + u_y = 1$ $\qquad u(x,0) = f(x)$. $\Rightarrow \begin{cases} x_0(s) = s \\ y_0(s) = 0 \\ u_0(s) = f(s) \end{cases}$

The characteristic equations are

$$\begin{cases} \dfrac{\partial x}{\partial \tau} = 1 & \Rightarrow \quad x = \tau + s \\ \dfrac{\partial y}{\partial \tau} = 1 & \Rightarrow \quad y = \tau \\ \dfrac{\partial u}{\partial \tau} = 1 & \Rightarrow \quad u = \tau + f(s) \end{cases}$$

so the projection of the characteristic curves in $(x-y)$ plane have equation

$y = x - s$ $\quad \rightarrow$ straight lines, slope 1, x-intercept s



Solution : $\begin{cases} u = \tau + f(s) \\ x = \tau + s \\ y = \tau \end{cases} \Rightarrow \begin{cases} \tau = y \\ s = x - y \\ \boxed{u = y + f(x-y)} \end{cases}$

2.2 $x u_x + (x+y) u_y = 1$ $\qquad u(1,y) = y \Rightarrow \begin{cases} x_0(s) = 1 \\ y_0(s) = s \\ u_0(s) = s \end{cases}$

Characteristic equations $\quad \begin{cases} \dfrac{\partial x}{\partial \tau} = x \\ \dfrac{\partial y}{\partial \tau} = x + y \\ \dfrac{\partial u}{\partial \tau} = 1 \end{cases}$

$$\Rightarrow \begin{cases} x = e^{\tau} \\ \dfrac{\partial y}{\partial \tau} - y = e^{\tau} \quad \Rightarrow \quad \dfrac{\partial}{\partial \tau}\left(y e^{-\tau}\right) = 1 \quad \Rightarrow \quad y e^{-\tau} - s = \tau \\ u = \tau + s \end{cases}$$

So
$$\begin{cases} x = e^{\tau} \\ y = e^{\tau}(\tau+s) \\ u = \tau+s \end{cases} \qquad \frac{y}{x} = \tau+s \quad \text{so} \quad \boxed{u = \frac{y}{x}}$$

$\Rightarrow$ the solution is not defined for $x=0$.

$\boxed{2.3}$ $\quad x u_x + y u_y = pu \qquad -\infty < x < +\infty \qquad -\infty < y < +\infty$

(a) Characteristic equations

$$\frac{\partial u}{\partial \tau} = pu \quad \Rightarrow \quad u = u_0(s) e^{p\tau}$$
$$\frac{\partial x}{\partial \tau} = x \quad \Rightarrow \quad x = x_0(s) e^{\tau}$$
$$\frac{\partial y}{\partial \tau} = y \quad \Rightarrow \quad y = y_0(s) e^{\tau}$$

(b) $p=4$, initial curve is $x^2+y^2=1$ with $u=1$

. first we parameterize it as $\begin{cases} x_0(s) = \cos(s) \\ y_0(s) = \sin(s) \\ u_0(s) = 1 \end{cases}$

then
$$\begin{cases} u = e^{4\tau} \\ x = \cos(s) e^{\tau} \\ y = \sin(s) e^{\tau} \end{cases}$$

$$\Rightarrow \quad x^2 + y^2 = e^{2\tau} \quad \Rightarrow \quad \boxed{u = (x^2+y^2)^2}$$

(c) $p=2$, initial curve is $u(x,0) = x^2$ for $x>0$

$$\begin{cases} x_0(s) = s \\ y_0(s) = 0 \\ u_0(s) = s^2 \end{cases}$$

$$\Rightarrow \begin{cases} u = s^2 e^{2\tau} \\ x = s e^{\tau} \\ y = 0 \end{cases} \quad \Rightarrow \quad \text{A possible solution: } u = x^2$$

$\boxed{2.7}$ $u_x + u_y = u^2 \qquad u(x,0) = 1 \Rightarrow \begin{cases} x_0(s) = s \\ y_0(s) = 0 \\ u_0(s) = 1 \end{cases}$

$$\frac{\partial x}{\partial \tau} = 1 \Rightarrow x = \tau + s$$

$$\frac{\partial y}{\partial \tau} = 1 \Rightarrow y = \tau$$

$$\frac{\partial u}{\partial \tau} = u^2 \Rightarrow \int \frac{\partial u}{u^2} = \tau + K$$

$$\Rightarrow -\frac{1}{u} = \tau + K \Rightarrow u = -\frac{1}{\tau + K}$$

when $\tau = 0$ $u = 1 \Rightarrow K = -1$

so $u = \dfrac{1}{1-\tau}$

So the characteristic curves are $\begin{cases} x = \tau + s \\ y = \tau \\ u = \dfrac{1}{1-\tau} \end{cases}$

and the solution is $u = \dfrac{1}{1-y}$

Note : the solution is independent of x – Could we have predicted this ?

Yes: the initial condition is independent of x and the solution has no explicit terms in x

$\boxed{2.11}$ $(y^2+u)u_x + yu_y = 0$ in $y > 0$ with $u = 0$ on $x = \dfrac{y^2}{2}$.

- initial condition curve : take $\begin{cases} x_0(s) = s^2/2 \\ y_0(s) = s \\ u_0(s) = 0 . \end{cases}$
- characteristic equations

$$\begin{cases} \dfrac{\partial x}{\partial \tau} = y^2 + u & (1) \\ \dfrac{\partial y}{\partial \tau} = y & (2) \\ \partial u / \partial \tau = 0 & (3) \end{cases}$$

From (3) $\Rightarrow$ $u = u_0(s) = 0$

So the solution appears to be 0 "everywhere".

Is this true?

Characteristics are ② $y = se^{\tau}$

① $\frac{\partial x}{\partial \tau} = s^2 e^{2\tau} \Rightarrow x = \frac{s^2}{2} e^{2\tau}$

so $x = \frac{y^2}{2}$

→ so the characteristics are all confounded with the initial condition curve.

⇒ it's a degenerate problem, with an ∞ of solutions

$\boxed{2.12}$ $\quad u_y + u^2 u_x = 0 \qquad x>0 \qquad u(x,0)=\sqrt{x}$

<u>characteristic equations</u> $\qquad \Rightarrow \begin{cases} x_0(s) = s \\ y_0(s) = 0 \\ u_0(s) = \sqrt{s} \end{cases}$ for $s>0$

$$\frac{\partial y}{\partial \tau} = 1 \quad \Rightarrow \quad y = \tau + y_0(s) = \tau$$

$$\frac{\partial x}{\partial \tau} = u^2 \quad \Rightarrow \quad \frac{\partial x}{\partial \tau} = \sqrt{s} \quad \Rightarrow \quad x = \sqrt{s}\,\tau + s$$

$$\frac{\partial u}{\partial \tau} = 0 \quad \Rightarrow \quad u = u_0(s) = \sqrt{s}$$

So $\quad x = \sqrt{s}\, y + s \qquad y = \dfrac{x-s}{\sqrt{s}}$

we can solve for $\sqrt{s}$ in terms of x and y:

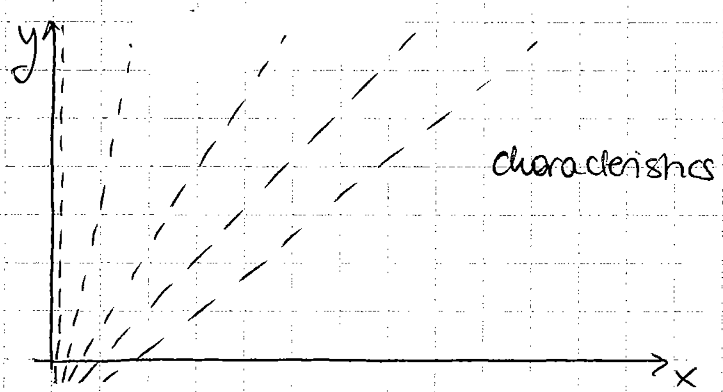
$$(\sqrt{s})^2 + \sqrt{s}\, y - x = 0$$

$$\sqrt{s} = \frac{-y \pm \sqrt{y^2 + 4x}}{2}$$

for $\sqrt{s} > 0$ we require the + solution.

so $u = \sqrt{s} = \dfrac{-y + \sqrt{y^2+4x}}{2}$

We see that the solution is defined everywhere for $x>0$



Note: the transversality condition reads in this case

$$\begin{vmatrix} s & 1 \\ 1 & 0 \end{vmatrix} = -1 \quad \Rightarrow \text{ nowhere } = 0.$$

$\boxed{2.13}$ $\quad uu_x + xu_y = 1$

$$x_0(s) = \frac{1}{2}s^2 + 1$$
$$y_0(s) = \frac{1}{6}s^3 + s$$
$$u_0(s) = s$$

$$\begin{cases} \dfrac{\partial x}{\partial \tau} = u & (2) \\ \dfrac{\partial y}{\partial \tau} = x & (3) \\ \dfrac{\partial u}{\partial \tau} = 1 \xrightarrow[(1)]{} u = \tau + s & \text{substitute in } (2) \end{cases}$$

$$\frac{\partial x}{\partial \tau} = \tau + s \quad \Rightarrow \quad x = \frac{\tau^2}{2} + s\tau + x_0(s)$$

$\rightarrow$ substitute in (3)

$$\frac{\partial y}{\partial \tau} = \frac{\tau^2}{2} + s\tau + x_0(s)$$
$$\Rightarrow \quad y = \frac{\tau^3}{6} + \frac{s\tau^2}{2} + x_0(s)\tau + y_0(s)$$

So
$$\begin{cases} x = \dfrac{\tau^2}{2} + s\tau + \dfrac{1}{2}s^2 + 1 = \dfrac{1}{2}(\tau+s)^2 + 1 \\ y = \dfrac{\tau^3}{6} + s\dfrac{\tau^2}{2} + \left(\dfrac{1}{2}s^2+1\right)\tau + \dfrac{1}{6}s^3 + s \\ u = \tau + s \end{cases}$$

So $\quad u = \sqrt{2(x-1)} \quad$ is a solution.

Are there other solutions?

Transversality condition:

$$\begin{vmatrix} s & s \\ \frac{1}{2}s^2+1 & \frac{1}{2}s^2+1 \end{vmatrix} = 0 \quad \Rightarrow$$

there can be either no solution or an ∞ of solutions.

Since we already found 1, we know there are an ∞ of solutions.

Write $\quad y = \frac{1}{6}\left[\tau^3 + 3s\tau^2 + 3s^2\tau + s^3\right] + \tau + s = \frac{1}{6}(\tau+s)^3 + (\tau+s)$

So we can also find a solution such that

$$y = \frac{1}{6}u^3 + u$$

$\boxed{2.14}$ $\quad x u_x + y u_y = \frac{1}{\cos u}$ $\qquad$ $x_0(s) = s^2$, $y_0(s) = \sin s$, $u_0(s) = 0$

(a) $\frac{\partial x}{\partial \tau} = x \quad \Rightarrow \quad x = s^2 e^{\tau}$

$\frac{\partial y}{\partial \tau} = y \quad \Rightarrow \quad y = \sin(s) e^{\tau}$

$\frac{\partial u}{\partial \tau} = \frac{1}{\cos u} \quad \Rightarrow \quad +\sin u = \tau + K$

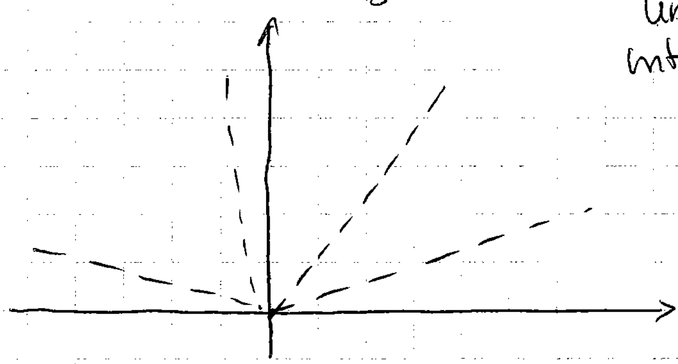
$+\sin(u_0(s)) = K$

$\Rightarrow \quad K = 0$

So
$$\begin{cases} x = s^2 e^{\tau} \\ y = \sin(s) e^{\tau} \\ u = \sin^{-1}(\tau) \end{cases}$$

Difficult to invert.

$y = \frac{\sin(s)}{s^2} x \quad \Rightarrow$ characteristics are straight lines out of the origin with slopes $\frac{\sin s}{s^2}$



$\boxed{2.16}$

$$xu_x + yu_y = -u \qquad \begin{cases} x_0(s) = \cos s \\ y_0(s) = \sin s \\ u_0(s) = 1 \end{cases} \quad 0 \leq s \leq \pi.$$

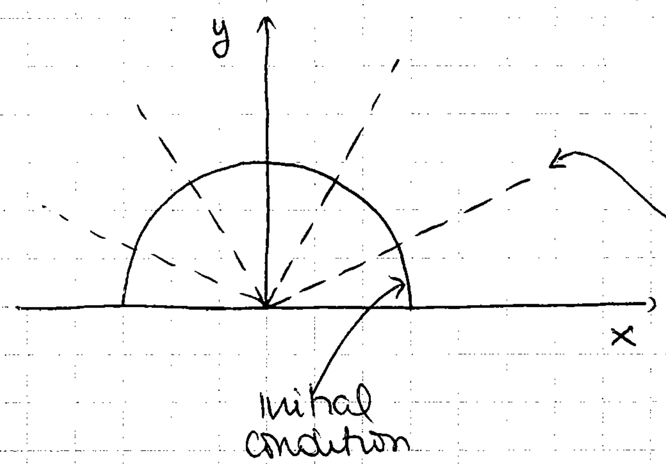
$$\frac{\partial x}{\partial \tau} = x \qquad x = \cos s\, e^{\tau}$$

$$\frac{\partial y}{\partial \tau} = y \qquad y = \sin s\, e^{\tau}$$

$$\frac{\partial u}{\partial \tau} = -u \qquad u = e^{-\tau}$$

so $x^2+y^2 = e^{2\tau}$ so $u = \dfrac{1}{\sqrt{x^2+y^2}}$

$\rightarrow$ not defined at $(0,0)$



Note characteristics are

$$y = \sin(s)\frac{x}{\cos(s)} = \tan(s)\,x$$

$\boxed{2.17}$ $\quad xu_x + u_y = 1$

. Find a characteristic curve through $(1,1,1)$

$$\frac{\partial u}{\partial \tau} = 1 \Rightarrow u = \tau + u_0(s)$$

$$\frac{\partial x}{\partial \tau} = x \Rightarrow x = x_0(s)e^{\tau}$$

$$\frac{\partial y}{\partial \tau} = 1 \Rightarrow y = \tau + y_0(s)$$

if $x_0(s) = 1$, $y_0(s) = 1$ and $u_0(s) = 1$ then

$$\begin{cases} x = e^{\tau} \\ y = \tau \\ u = \tau \end{cases}$$ is the parametric equation for the characteristic curve.

(b) if $u(x,0) = \sin x$ then

$$x_0(s) = s$$
$$y_0(s) = 0$$
$$u_0(s) = \sin s$$

$$\Rightarrow \begin{cases} x = se^{\tau} \\ y = \tau \\ u = \tau + \sin(s) \end{cases}$$

so $x = se^{y} \Rightarrow s = xe^{-y}$

so $u = y + \sin\left(xe^{-y}\right)$

the solution is defined for all x and y

$\boxed{2.18}$ $uu_x + u_y = -\frac{1}{2}u$ $\qquad u(x,2x) = x^2$

$$\begin{cases} x_0(s) = s \\ y_0(s) = 2s \\ u_0(s) = s^2 \end{cases}$$

so ③ $\dfrac{\partial x}{\partial \tau} = u \Rightarrow \dfrac{\partial x}{\partial \tau} = s^2 e^{-\frac{1}{2}\tau} \Rightarrow x = -2s^2 e^{-\frac{1}{2}\tau} + s + 2s^2$

① $\dfrac{\partial y}{\partial \tau} = 1 \Rightarrow y = \tau + 2s$

② $\dfrac{\partial u}{\partial \tau} = -\dfrac{1}{2}u \Rightarrow u = s^2 e^{-\frac{1}{2}\tau}$

So $\tau = y - 2s \Rightarrow x = -2s^2 e^{-\frac{1}{2}(y-2s)} + s + 2s^2$

Possible but difficult to invert for s

However we can identify the condition for existence/uniqueness of solutions from the transversality condition

as $a(x_0, y_0, u_0) = s^2 \qquad \dfrac{dx_0}{ds} = 1$

$b(x_0, y_0, u_0) = 1 \qquad \dfrac{dy_0}{ds} = 2$

$$\Rightarrow \begin{vmatrix} a & dx_0/ds \\ b & dy_0/ds \end{vmatrix} = 2s^2 - 1 \Rightarrow \text{problem when } s = \pm\sqrt{\tfrac{1}{2}}.$$

$\boxed{2.19}$

$$x^2 u_x + y^2 u_y = u^2 \qquad u(x,2x) = x^2 \Rightarrow \begin{cases} x_0(s) = s \\ y_0(s) = 2s \\ u_0(s) = s^2 \end{cases}$$

- Transversality condition

$$\begin{vmatrix} s^2 & 1 \\ 4s^2 & 2 \end{vmatrix} = 2s^2 - 4s^2 = -2s^2 \neq 0 \text{ unless } s=0$$

- Solution

$$\frac{\partial x}{\partial \tau} = x^2 \Rightarrow -\frac{1}{x} = \tau + K_1 \Rightarrow K_1 = -\frac{1}{s}$$

$$\frac{\partial y}{\partial \tau} = y^2 \qquad -\frac{1}{y} = \tau + K_2 \Rightarrow K_2 = -\frac{1}{2s}$$

$$\frac{\partial u}{\partial \tau} = u^2 \qquad -\frac{1}{u} = \tau + K_3 \qquad K_3 = -\frac{1}{s^2}$$

So
$$\begin{cases} x = -\dfrac{1}{\tau+K_1} = -\dfrac{1}{\tau - 1/s} = \dfrac{s}{1-\tau s} & (1) \\ y = -\dfrac{1}{\tau+K_2} = -\dfrac{1}{\tau - 1/2s} = \dfrac{2s}{1-2\tau s} & (2) \\ u = -\dfrac{1}{\tau+K_3} = -\dfrac{1}{\tau - 1/s^2} = \dfrac{s^2}{1-s^2\tau} & (3) \end{cases}$$

so from (1) $\quad (1-\tau s)x = s \Rightarrow s = \dfrac{x}{1+\tau x}$

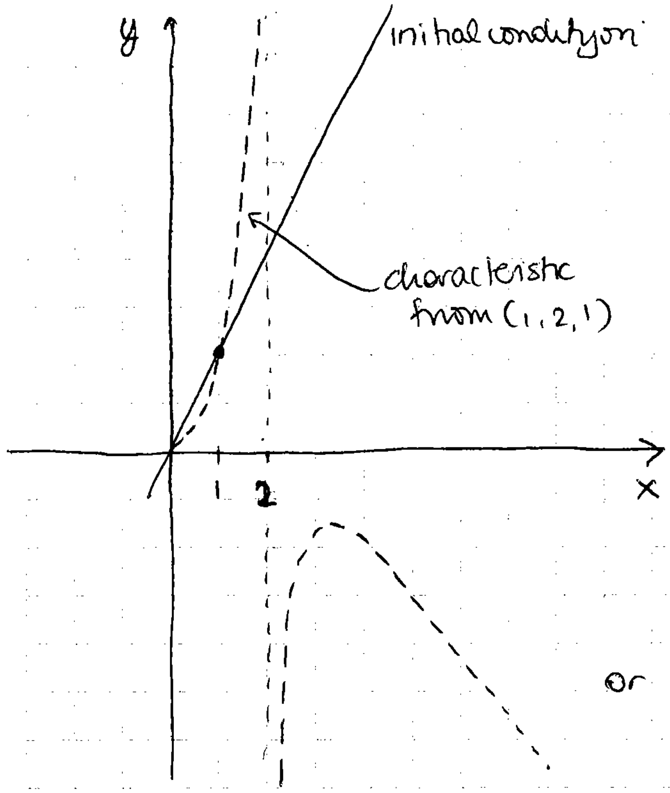
from (2) $\quad y = 2\dfrac{x}{1+\tau x}\dfrac{1}{1-\dfrac{2\tau x}{1+\tau x}} = \dfrac{2x}{1+\tau x - 2\tau x} = \dfrac{2x}{1-\tau x}$

so $\quad y(1-\tau x) = 2x \Rightarrow \tau = \dfrac{1-\frac{2x}{y}}{x} = \dfrac{1}{x} - \dfrac{2}{y}$

and
$$s = \frac{x}{1+1-\frac{2x}{y}} = \frac{x}{2(1-\frac{x}{y})}$$

So
$$u = \frac{x^2}{4(1-\frac{x}{y})^2}\,\frac{1}{1-\frac{x^2}{4(1-\frac{x}{y})^2}\cdot\frac{1}{x}(1-\frac{2x}{y})} = \frac{x^2}{4(1-\frac{x}{y})^2 - x(1-\frac{2x}{y})}$$

$$= \frac{x^2}{4 - 8\frac{x}{y} + \frac{4x^2}{y^2} - x + \frac{2x^2}{y}} \Rightarrow \text{solution undefined when } 4(1-\tfrac{x}{y})^2 - x(1-\tfrac{2x}{y}) = 0$$



characteristic that starts at $(0,0,0) \rightarrow s = 0 \quad \begin{cases} x = 0 \\ y = 0 \end{cases}$

$\Rightarrow$ a point

characteristic that starts at $(1,2,1) \rightarrow s = 1$

$$\begin{cases} x = \dfrac{1}{1-\tau} \\[2mm] y = \dfrac{2}{1-2\tau} \end{cases}$$

or $\tau = 1 - \frac{1}{x} \Rightarrow y = \dfrac{2}{1-2(1-\frac{1}{x})} = \dfrac{2}{\frac{2}{x} - 1}$

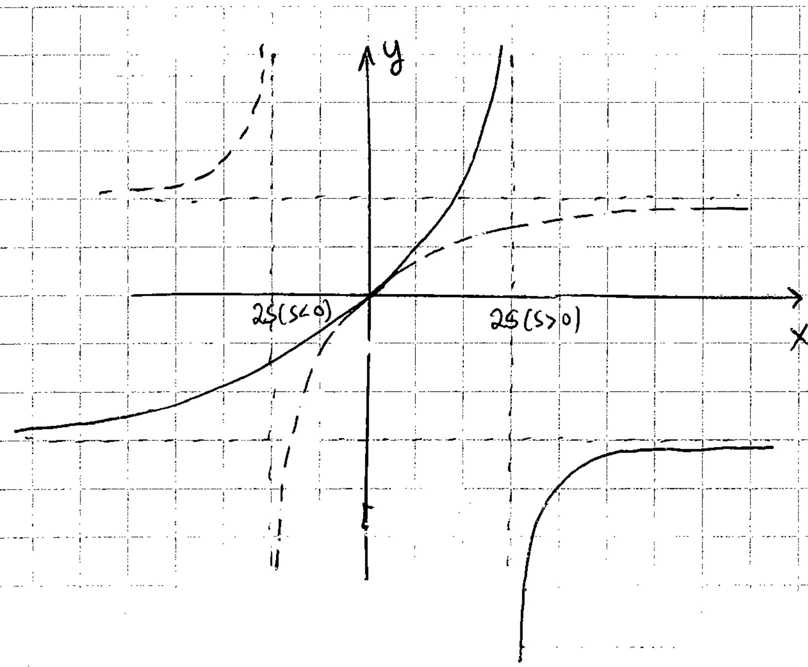
$$= \frac{2x}{2-x}$$

(a) Characteristics are given by

$$1 - \tau s = \frac{s}{x} \Rightarrow \tau s = 1 - \frac{s}{x} \Rightarrow$$

$$y = \frac{2s}{1 - 2\left(1 - \frac{s}{x}\right)} \Rightarrow y = \frac{2s}{\frac{2s}{x} - 1} = \frac{2sx}{2s - x}$$

The characteristics all look like



—— case $s > 0$

---- case $s < 0$

$\rightarrow$ problem at
$x = 0$
$y = 0$
where all characteristics cross

$\boxed{2.20}$ $\quad y u_x - u u_y = x$

Characteristic curves

$$\begin{cases} \frac{\partial x}{\partial \tau} = y \\ \frac{\partial y}{\partial \tau} = -u \\ \frac{\partial u}{\partial \tau} = x \end{cases} \Rightarrow \begin{cases} \frac{\partial^3 x}{\partial \tau^3} = -x \\ \frac{\partial^3 y}{\partial \tau^3} = -y \\ \frac{\partial^3 u}{\partial \tau^3} = -u \end{cases}$$

Try solutions of the form $x = \tilde{x} e^{\lambda \tau} \Rightarrow \lambda^3 = -1$

$$\Rightarrow \begin{cases} \lambda = -1 \\ \lambda = e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2} \\ \lambda = e^{-i\pi/3} = \frac{1}{2} - i\frac{\sqrt{3}}{2} \end{cases}$$

so by a linear combination of the above we find

$$x = Ae^{-\tau} + Be^{\frac{1}{2}\tau + i\frac{\sqrt{3}}{2}\tau} + Ce^{\frac{1}{2}\tau - i\frac{\sqrt{3}}{2}\tau}$$

or $\quad x = Ae^{-\tau} + e^{\frac{1}{2}\tau}\left(B'\cos\left(\frac{\sqrt{3}}{2}\tau\right) + C'\sin\left(\frac{\sqrt{3}}{2}\tau\right)\right)$

$$y = \frac{\partial x}{\partial \tau} = -Ae^{-\tau} + \frac{1}{2}e^{\frac{1}{2}\tau}\left(B'\cos\left(\frac{\sqrt{3}}{2}\tau\right) + C'\sin\left(\frac{\sqrt{3}}{2}\tau\right)\right)$$
$$+ e^{\frac{1}{2}\tau}\left(-\frac{\sqrt{3}}{2}B'\sin\left(\frac{\sqrt{3}}{2}\tau\right) + \frac{\sqrt{3}}{2}C'\cos\left(\frac{\sqrt{3}}{2}\right)\tau\right)$$
$$= -Ae^{-\tau} + e^{\frac{1}{2}\tau}\left[\left(\frac{1}{2}B' + \frac{\sqrt{3}}{2}C'\right)\cos\left(\frac{\sqrt{3}}{2}\tau\right) + \left(\frac{1}{2}C' - \frac{\sqrt{3}}{2}B'\right)\sin\left(\frac{\sqrt{3}}{2}\tau\right)\right]$$

$$u = -\frac{\partial y}{\partial \tau} = -Ae^{-\tau} - \frac{1}{2}e^{\frac{1}{2}\tau}\left[\quad --- \quad\right]$$
$$- e^{\frac{1}{2}\tau}\left[-\frac{\sqrt{3}}{2}\left(\frac{1}{2}B' + \frac{\sqrt{3}}{2}C'\right)\sin\left(\frac{\sqrt{3}}{2}\tau\right) + \frac{\sqrt{3}}{2}\left(\frac{1}{2}C' - \frac{\sqrt{3}}{2}B'\right)\cos()\right]$$
$$= -Ae^{-\tau} - \frac{e^{\frac{1}{2}\tau}}{4}\left[\left(B' + 2\sqrt{3}C' - 3B'\right)\cos\left(\frac{\sqrt{3}}{2}\tau\right) + \left(C' - 2\sqrt{3}B' - 3C'\right)\sin\left(\frac{\sqrt{3}}{2}\tau\right)\right)$$
$$= -Ae^{-\tau} + e^{\frac{1}{2}\tau}\left[\left(\frac{B'}{2} - \frac{\sqrt{3}}{2}C'\right)\cos\left(\frac{\sqrt{3}}{2}\tau\right) + \left(\frac{\sqrt{3}}{2}B' + \frac{C'}{2}\right)\sin\left(\frac{\sqrt{3}}{2}\tau\right)\right]$$

To fit the initial conditions x_0, y_0 and u_0, the constants A, B' and C' must satisfy

$$A + B' = x_0$$

$$-A + \left(\frac{1}{2}B' + \frac{\sqrt{3}}{2}C'\right) = y_0$$

$$-A + \left(\frac{B'}{2} - \frac{\sqrt{3}}{2}C'\right) = u_0$$

so $\sqrt{3}C' = y_0 - u_0 \qquad C' = \frac{y_0 - u_0}{\sqrt{3}}$

$$\frac{3}{2}B' + \frac{\cancel{\sqrt{3}}}{2}\left(\frac{y_0 - u_0}{\cancel{\sqrt{3}}}\right) = x_0 + y_0$$

$$\Rightarrow \frac{3}{2}B' = x_0 + y_0 - \frac{y_0 - u_0}{2} = \frac{2x_0 + y_0 + u_0}{2}$$

$$B' = \frac{2x_0 + y_0 + u_0}{3}$$

$$A = x_0 - B' = x_0 - \frac{2x_0 + y_0 + u_0}{3}$$

$$A = \frac{x_0 - y_0 - u_0}{3}$$

So in the case where $u(s,s) = -2s$ then $x_0 = s$, $y_0 = s$, $u_0 = -2s$

so

$$C' = \frac{3s}{\sqrt{3}} = \sqrt{3}\,s$$

$$B' = \frac{s}{3}$$

$$A' = \frac{2s}{3}$$

$$B' + \sqrt{3}C' = \left(\frac{1}{3} + \sqrt{3}\sqrt{3}\right)s = \left(\frac{1}{3} + 3\right)s$$

$$C' - \sqrt{3}B' = \left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)s$$

so

$$\begin{cases} x = \frac{2s}{3}e^{-\tau} + e^{\frac{1}{2}\tau}s\left(\frac{1}{3}\cos\left(\frac{\sqrt{3}}{2}\tau\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\tau\right)\right) \\ y = -\frac{2s}{3}e^{-\tau} + e^{\frac{1}{2}\tau}\frac{s}{2}\left(\left(\frac{1}{3}+3\right)\cos\left(\frac{\sqrt{3}}{2}\tau\right) + \left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)\sin\left(\frac{\sqrt{3}}{2}\tau\right)\right) \\ u = -\frac{2s}{3}e^{-\tau} + \frac{s}{2}e^{\frac{1}{2}\tau}\left(\left(\frac{1}{3} - 3\right)\cos\left(\frac{\sqrt{3}}{2}\tau\right) + \left(\frac{1}{\sqrt{3}} + \sqrt{3}\right)\sin\left(\frac{\sqrt{3}}{2}\tau\right)\right) \end{cases}$$

$B - \sqrt{3}C'$

$\frac{1}{3} - 3$