Take Home Midterm

Near-complete and correct answers for 3 out of the 4 problems will earn you an A. To get an A+ you will need to answer the 4 problems correctly. You need to justify all your answers. Answers without justifications will be counted as wrong.

Problem 1: 2D transport

Let's approximate the coast of North California as a straight North-South line, and use a coordinate system where $x = 0$ denotes the coast ($x > 0$ is inland, $x < 0$ is the Ocean). In what follows, the unit length is a mile, and the unit time is an hour. San Francisco is located at the point $(0, 0)$, and Santa Cruz at the point $(0, -50)$.

An oil tanker accidentally releases a circular patch of oil, of 1-mile radius, centered 10 miles off the coast from San Francisco. The initial oil concentration field, at time $t = 0$, is

$$c(x, y, t = 0) = 1 \text{ if } (x + 10)^2 + y^2 < 1$$
$$c(x, y, t = 0) = 0 \text{ if } (x + 10)^2 + y^2 > 1$$

The oil patch is advected by the Southward current as well as by East-West tidal currents, following the equation:

$$\frac{\partial c}{\partial t} + \nabla \cdot (c\mathbf{v}) = 0$$

where the velocity field is given by $\mathbf{v} = (-2\cos(2\pi t/12), -1)$.

(1) Solve the PDE above with the given initial condition, and show that the patch remains circular. What is the trajectory of its center?
(2) Will surfers in Santa Cruz have to get out of the water to avoid the spill (you may assume surfers stay very close to the coast, at $x = 0$)? If yes, when? If no, what is the furthest distance out to sea that boats will be able to go and still avoid the patch, while the oil patch is passing by?

Problem 2: Canonical forms

Textbook p. 74, problem 3.6

Problem 3: Molecular diffusion

Consider a 1-D rod of unit length, thermally insulated on all sides (Note: a thermally insulating boundary is such that $\nabla T \cdot \mathbf{n} = 0$ on the boundary, where n is the normal to the boundary).

At time $t = 0$, the left-half of the rod ($x \in [0, 1/2]$) is at uniform temperature $T_1 = 0$ and the right-half of the rod ($x \in [1/2, 1]$) is at uniform temperature $T_2 = 1$. The thermal diffusion coefficient in the rod is $k = 1$.

(1) Construct the mathematical model associated with this problem.
(2) What is the steady-state solution of your model (for $t \to \infty$)?
(3) Find the analytical solution to the initial value problem.
(4) Using a computer, plot the solution at times $t = 0$, $t = 0.01$, $t = 0.1$, $t = 1$. Briefly describe what method you are using to plot the solution.

Problem 4: The Beam equation

Read the attached handout carefully, then answer the following questions:

(1) What equation do the eigenfrequencies of the cantilever beam satisfy?
(2) Give numerical values for the eigenfrequencies of the first five modes (explain which method you are using to get these values).
