what is the subsequent motion of the string if it is suddenly released?

6. Solve the damped vibrating-string problem

PDE	$u_{tt} = \alpha^2 u_{xx} - \beta u_t \quad 0 < x < 1$	$0 < t < \infty$
BCs	$\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}$	$0 < t < \infty$
ICs	$\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases}$	$0 \leq x \leq 1$

Does the solution seem reasonable? Does it satisfy the above PDE, BCs, and ICs?

7. How would you solve the *nonhomogeneous* PDE with given boundary and initial conditions

PDE	$u_{tt} = \alpha^2 u_{xx} + Kx \quad 0 < x < 1$	$0 < t < \infty$
BCs	$\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}$	$0 < t < \infty$
ICs	$\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases}$	$0 \leq x \leq 1$

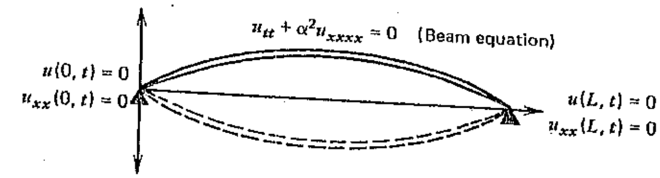
OTHER READING

Advanced Engineering Mathematics by C. Wylie. McGraw-Hill, 1970. A very readable text that contains many interesting examples; see in particular Chapter 7.

LESSON 21

The Vibrating Beam (Fourth-Order PDE)

PURPOSE OF LESSON: To illustrate how higher-order PDEs come about in the study of vibrating-beam problems and to solve the problem of a vibrating beam with simply supported ends by separation of variables. It is also pointed out how the vibrations of the beam *compare* with the vibrations of the violin string.



Simply supported beam

The major difference between the transverse vibrations of a violin string and the transverse vibrations of a thin beam is that the beam offers resistance to bending. Without going into the mechanics of thin beams, we can show that this resistance is responsible for changing the wave equation to the fourth-order beam equation

$$u_{tt} = -\alpha^2 u_{xxxx} \tag{21.1}$$

where

$\alpha^2 = K/\rho$

K = rigidity constant (the larger K, the more rigid the beam and the faster the vibrations)

ρ = linear density of the beam (mass/unit length).

The derivation of this equation can be found in reference 1 of Other Reading. Since this is the first time the reader has seen an application of PDEs higher than second order in this text, it will be useful to solve a typical vibrating-beam problem. Later, we will talk about other types of beam problems

The Simply Supported Beam

Consider the small vibrations of a thin beam whose ends are simply fastened to two foundations. By "simply fastened," we mean that the ends of the beam are held stationary, but the slopes at the end points can move (the beam is held by a pin-type arrangement, Figure 21.1).

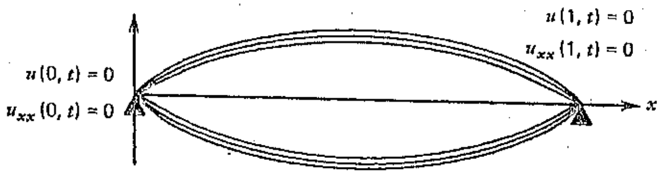


FIGURE 21.1 A simply supported beam.

It seems clear that the BCs at the ends of the beam should be

$$u(0,t) = 0$$
$$u(1,t) = 0$$

but what *isn't* so obvious is that the two BCs

$$u_{xx}(0,t) = 0$$
$$u_{xx}(1,t) = 0$$

also hold at the two ends. Using the theory of thin beams (see reference 1 of Other Reading), we can show that the *bending moment* of the beam is represented by u_{xx} and a simply fastened beam should have zero moments at the end points. Hence, the vibrating beam in Figure 21.1 can be described by the IBVP (α is set equal to one for simplicity)

PDE $\quad u_{tt} = -u_{xxxx} \qquad 0 < x < 1 \qquad 0 < t < \infty$

(21.2) BCs $\quad \begin{cases} u(0,t) = 0 \\ u_{xx}(0,t) = 0 \\ u(1,t) = 0 \\ u_{xx}(1,t) = 0 \end{cases} \qquad 0 < t < \infty$

ICs $\quad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \qquad 0 \le x \le 1$

To solve this problem, we use the separation of variables method and look for arbitrary periodic solutions; that is, vibrations of the form

$$u(x,t) = X(x)[A \sin(\omega t) + B \cos(\omega t)] \tag{21.3}$$

Note that by choosing the solution in the form (21.3), we are essentially saying that the separation constant in the separation-of-variables method has been chosen to be negative.

We now substitute equation (21.3) into the beam equation to get the ODE in $X(x)$.

$$X^{iv} - \omega^2 X = 0$$

which has the general solution

$$X(x) = C \cos\sqrt{\omega}x + D \sin\sqrt{\omega}x + E \cosh\sqrt{\omega}x + F \sinh\sqrt{\omega}x$$

To find the constants C, D, E, and F, we substitute this expression into the BCs, giving

$$\left.\begin{aligned} u(0,t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0 \quad &\Rightarrow C + E = 0 \\ u_{xx}(0,t) = 0 \Rightarrow X''(0)T(t) = 0 \Rightarrow X''(0) = 0 \quad &\Rightarrow -C + E = 0 \end{aligned}\right\} \Rightarrow C = E = 0$$

$$u(1,t) = 0 \Rightarrow D \sin\sqrt{\omega} + F \sinh\sqrt{\omega} = 0$$
$$u_{xx}(1,t) = 0 \Rightarrow -D \sin\sqrt{\omega} + F \sinh\sqrt{\omega} = 0$$

From these last two equations, we arrive at the expressions

$$F \sinh\sqrt{\omega} = 0$$
$$D \sin\sqrt{\omega} = 0$$

from which we can conclude

$$F = 0$$
$$\sin\sqrt{\omega} = 0 \Rightarrow \omega = (n\pi)^2 \qquad n = 1, 2, \ldots$$

In other words, the *natural frequencies* of the simply supported beam are

$$\omega_n = (n\pi)^2$$

and the *fundamental solutions* u_n (solutions of the PDE and BCs) are

$$u_n(x,t) = X_n(x)T_n(t) = [a_n \sin(n\pi)^2 t + b_n \cos(n\pi)^2 t] \sin(n\pi x)$$

Now, since the PDE and BCs are linear and homogeneous, we can conclude that the sum

$$u(x,t) = \sum_{n=1}^{\infty} [a_n \sin (n\pi)^2 t + b_n \cos (n\pi)^2 t] \sin (n\pi x) \tag{21.4}$$

also satisfies the PDE and BCs. Hence, all that remains to do is choose the constants a_n and b_n in such a way that the ICs are satisfied. Substituting equation (21.4) into the ICs gives us

$$\begin{aligned} u(x,0) &= f(x) = \sum_{n=1}^{\infty} b_n \sin (n\pi x) \\ u_t(x,0) &= g(x) = \sum_{n=1}^{\infty} (n\pi)^2 a_n \sin (n\pi x) \end{aligned} \tag{21.5}$$

and using the fact that the family $\{\sin (n\pi x)\}$ is orthogonal on the interval [0,1] gives us

$$\begin{aligned} a_n &= \frac{2}{(n\pi)^2} \int_0^1 g(x) \sin (n\pi x)\, dx \\ b_n &= 2\int_0^1 f(x) \sin (n\pi x)\, dx \end{aligned} \tag{21.6}$$

Hence, the solution is given by (21.4), and a_n and b_n are given by (21.6).

In order for the reader to understand this problem, we present a simple example.

Sample Vibrating Beam

Consider the simply supported beam shown in Figure 21.2 with ICs

$$\begin{aligned} u(x,0) &= \sin (\pi x) + 0.5 \sin (3\pi x) \\ u_t(x,0) &= 0 \end{aligned}$$

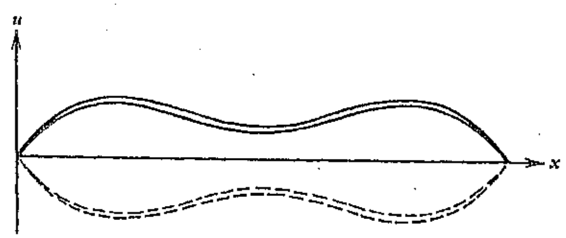


FIGURE 21.2 Simple vibrations of a simply supported beam.

We could find the solution by substituting the values of $f(x)$ and $g(x)$ into equation (21.6), but it seems easier to look at equations (21.5) and simply make the observation that

$$\begin{aligned} a_n &= 0 \qquad \text{for all } n = 1, 2, \ldots \\ b_1 &= 1 \\ b_2 &= 0 \\ b_3 &= 0.5 \\ b_n &= 0 \qquad n = 4, 5, \ldots \end{aligned}$$

Hence, the solution is

$$u(x,t) = \cos (\pi^2 t) \sin (\pi x) + 0.5 \cos (9\pi^2 t) \sin (3\pi x)$$

It is interesting to see how this solution compares with the vibrating string with the same ICs. If we look back to Lesson 20, we find that the solution to the vibrating-string problem is given by

$$u(x,t) = \cos (\pi t) \sin (\pi x) + 0.5 \cos (3\pi t) \sin (3\pi x)$$

In other words, the *vibrating beam* vibrates at higher frequencies than does the *vibrating string*. It would be interesting for the reader to imagine just how each of these vibrations looks. Note, however, that both higher frequencies are integer multiples of the fundamental frequencies.

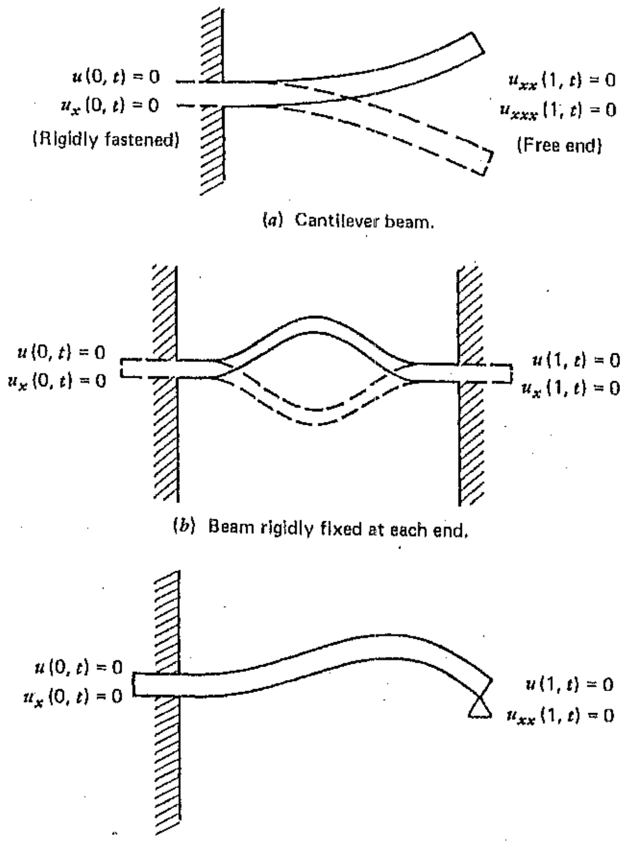
NOTES

1. Beams are generally fastened in one of three ways
   (a) Free (unfastened)
   (b) Simply fastened
   (c) Rigidly fastened

   Some sketches are given in Figure 21.3 along with their BCs.
2. Another important vibrating-beam problem is the *cantilever-beam problem* shown in Figure 21.3. The solution to this vibrating beam is not the usual sum of products of sines and cosines, but due to the nonstandard BCs,

$$\begin{aligned} u(0,t) &= 0 \\ u_x(0,t) &= 0 \\ u_{xx}(1,t) &= 0 \\ u_{xxx}(1,t) &= 0 \end{aligned}$$

we arrive at the more complicated solution



(a) Cantilever beam.

(b) Beam rigidly fixed at each end.

(c) Beam rigidly fastened at left; simply fastened at right.

FIGURE 21.3a–21.3c Typical beam problems.

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)\,[a_n \sin(\omega_n t) + b_n \cos(\omega_n t)]$$

where the eigenfunctions (basic shapes of vibrations) are given by linear combinations of sines, cosines, hyperbolic sines, and hyperbolic cosines.

PROBLEMS

1. Solve the *cantilever-beam* problem

$$\text{PDE} \qquad u_{tt} + u_{xxxx} = 0 \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$\text{BCs} \quad \begin{cases} u(0,t) = 0 \\ u_x(0,t) = 0 \\ u_{xx}(1,t) = 0 \\ u_{xxx}(1,t) = 0 \end{cases} \qquad 0 < t < \infty$$

$$\text{ICs} \quad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \qquad 0 \leqslant x \leqslant 1$$

HINT Although the eigenfunctions $X_n(x)$ in this problem are not the usu sine functions, we can still use the Sturm-Liouville theory to say that t eigenfunctions are orthogonal on [0,1].

2. What is the solution to the simply supported beam (at both ends) with IC

$$\begin{aligned} u(x,0) &= \sin(\pi x) \\ u_t(x,0) &= \sin(\pi x) \end{aligned} \qquad 0 \leqslant x \leqslant 1$$

3. What is the solution to the simply supported beam problem with ICs

$$\begin{aligned} u(x,0) &= 1 - x^2 \\ u_t(x,0) &= 0 \end{aligned} \qquad 0 \leqslant x \leqslant 1$$

4. Let the left end ($x = 0$) of a beam be rigidly fastened to a wall and let th right end ($x = 1$) be simply fastened according to the BCs shown in Figur 21.3. Solve the beam problem with these BCs and tell how to find the *natur frequencies of vibration* of this beam. Knowing the natural frequencies o the beam is important, since various kinds of inputs of the same frequenc can give rise to resonance.