MIDTERM 2009 ANSWERS.

Problem 1 [30]

$$\frac{\partial c}{\partial t} + \nabla\cdot(\underline{v}c) = 0 \qquad \underline{v} = \left(-2\cos\left(\frac{2\pi t}{12}\right), -1\right)$$

$$c(x,y,0) = \begin{cases} 1 \text{ if } (x+10)^2+y^2<1 \\ 0 \text{ if } (x+10)^2+y^2>1. \end{cases}$$

1. To solve the PDE

- express in cartesian coordinates

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x}\left(-2\cos\left(\frac{2\pi t}{12}\right)c\right) + \frac{\partial}{\partial y}\left(-c\right) = 0$$

$$\Rightarrow \quad \frac{\partial c}{\partial t} - 2\cos\left(\frac{2\pi t}{12}\right)\frac{\partial c}{\partial x} - \frac{\partial c}{\partial y} = 0$$

- Construct the initial condition "curve":

$$x_0(s_1,s_2) = s_1$$
$$y_0(s_1,s_2) = s_2$$
$$t_0(s_1,s_2) = 0$$
$$\begin{cases} c_0(s_1,s_2) = 1 \text{ if } (s_1+10)^2+s_2^2<1 \\ c_0(s_1,s_2) = 0 \text{ if } (s_1+10)^2+s_2^2>1 \end{cases}$$

- Characteristic equations:

$$\frac{\partial t}{\partial \tau} = 1 \quad \Rightarrow \quad t = \tau + t_0(s_1,s_2) = \tau$$

$$\frac{\partial x}{\partial \tau} = -2\cos\left(\frac{2\pi t}{12}\right) \quad \Rightarrow \quad \frac{\partial x}{\partial \tau} = -2\cos\left(\frac{2\pi \tau}{12}\right)$$

$$\Rightarrow x = -\frac{12}{\pi}\sin\left(\frac{2\pi\tau}{12}\right) + x_0(s_1,s_2)$$

$$= s_1 - \frac{12}{\pi}\sin\left(\frac{2\pi\tau}{12}\right) +$$

$$\frac{\partial y}{\partial \tau} = -1 \quad \Rightarrow \quad y = -\tau + y_0(s_1,s_2)$$

$$= -\tau + s_2$$

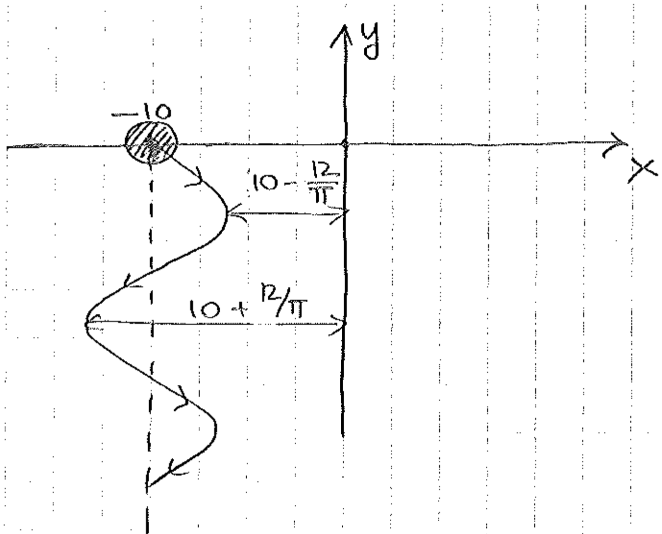
$$\frac{\partial c}{\partial \tau} = 0 \quad \Rightarrow \quad c = c_0(s_1,s_2)$$

$$\Rightarrow c(x,y,t) = c_0\left[x + \frac{12}{\pi}\sin\left(\frac{2\pi t}{12}\right), y+t\right]$$

$$\Rightarrow \quad c(x,y;t) = \begin{cases} 1 & \text{if } \left[x+\frac{12}{\pi}\sin\left(\frac{2\pi t}{12}\right)+10\right]^2+(y+t)^2<1 \\ 0 & \text{if } \left[x+\frac{12}{\pi}\sin\left(\frac{2\pi t}{12}\right)+10\right]^2+(y+t)^2>1 . \end{cases}$$

This formula implies that the shape of the patch remains the same, with a radius 1 but the center has a trajectory

$$\begin{cases} x_c(t) = -\frac{12}{\pi}\sin\left(\frac{2\pi t}{12}\right)-10 \\ y_c(t) = -t \end{cases}$$



Note: $\frac{12}{\pi} \simeq 3.8$

$\Rightarrow$ The patch never reaches the coast

2. The patch passes by Santa Cruz at time $t = 50$ (set $y_c = -50$, solve for t)

$\Rightarrow$ at that time $x_c(50) = -\frac{12}{\pi}\sin\left(\frac{100\pi}{12}\right)-10$

$\cong -13.3$ miles

$\Rightarrow$ The center of the patch will be $\simeq$ 13.3 miles offshore so boats may go out to about 12.3 miles.

Surfers do not need to get out

Note: One could also refine the calculation for 2. to include other effects, but this is OK.

Problem 2 [30]

$$u_{xx} - (1+y^2)^2 u_{yy} - 2y(1+y^2)u_y = 0$$

1. Canonical form

$$\delta = +(1+y^2)^2 > 0 \Rightarrow \text{hyperbolic equation}$$

$$\frac{dy}{dx} = \pm\sqrt{(1+y^2)^2} = \pm(1+y^2) \quad (\text{since } 1+y^2>0 \text{ always})$$

$$\Rightarrow \quad \frac{dy}{1+y^2} = \pm\, dx \Rightarrow \begin{cases} \operatorname{atan}(y) = x + \xi \\ \operatorname{atan}(y) = -x + \eta \end{cases}$$

$$\Rightarrow \quad \xi = \operatorname{atan}(y) - x$$
$$\eta = \operatorname{atan}(y) + x$$

$$\xi_x = -1 \qquad \eta_x = 1$$
$$\xi_y = \frac{1}{1+y^2} \qquad \eta_y = \frac{1}{1+y^2}$$
$$\xi_{yy} = \frac{-2y}{(1+y^2)^2} \qquad \eta_{yy} = \frac{-2y}{(1+y^2)^2}$$

$$u_{xx} = (-1)^2 u_{\xi\xi} + 2(-1)(1)u_{\xi\eta} + (1)^2 u_{\eta\eta}$$
$$u_{yy} = \left(\frac{1}{1+y^2}\right)^2 u_{\xi\xi} + 2\left(\frac{1}{1+y^2}\right)^2 u_{\xi\eta} + \left(\frac{1}{1+y^2}\right)^2 u_{\eta\eta} - \frac{2y}{(1+y^2)^2}u_\xi - \frac{2y}{(1+y^2)^2}u_\eta$$

$$u_y = \frac{1}{1+y^2}u_\xi + \frac{1}{1+y^2}u_\eta$$

$\Rightarrow$ The PDE becomes

$$\cancel{u_{\xi\xi}} - 2u_{\xi\eta} + \cancel{u_{\eta\eta}} - \cancel{u_{\xi\xi}} - 2u_{\xi\eta} - \cancel{u_{\eta\eta}} + \cancel{2y u_\xi} + \cancel{2y u_\eta} - \cancel{2y(u_\xi + u_\eta)} = 0$$

$$\Rightarrow \quad -4u_{\xi\eta} = 0 \Rightarrow \boxed{u_{\xi\eta} = 0}$$

(b) This has solutions

$$v(\xi,\eta) = F(\xi) + G(\eta)$$

$$\Rightarrow v(x,y) = F(\operatorname{atan}(y) - x) + G(\operatorname{atan}(y) + x).$$

(c) $u(x,0) = g(x) \qquad u_y(x,0) = f(x)$

Note: $\dfrac{\partial v}{\partial y} = \dfrac{1}{1+y^2}\left[F'(\operatorname{atan}(y) - x) + G'(\operatorname{atan}(y)+x)\right]$

$$u(x,0) = g(x) \Rightarrow F(-x) + G(x) = g(x) \qquad (1)$$

$$u_y(x,0) = f(x) \Rightarrow F'(-x) + G'(x) = f(x) \qquad (2)$$

Take the derivative of (1) $\Rightarrow$

$$-F'(-x) + G'(x) = g'(x)$$

Then add to (2) $\Rightarrow$

$$2G'(x) = f(x) + g'(x)$$

$$G(x) - G(0) = \frac{1}{2}\left[\int_0^x f(x')dx' + g(x) - g(0)\right].$$

So

$$F(-x) = g(x) - G(x)$$

$$= -\frac{1}{2}\int_0^x f(x')dx' + \frac{1}{2}g(x) + \frac{1}{2}g(0) - G(0)$$

$$\Rightarrow F(x) = \frac{1}{2}\int_{-x}^0 f(x')dx' + \frac{1}{2}g(-x) + \frac{1}{2}g(0) - G(0)$$

(d) $f(x) = -2x \qquad g(x) = x$

$$G(x) = G(0) + \frac{1}{2}(-x^2 + x) = G(0) + \frac{x - x^2}{2}$$

$$F(x) = \frac{1}{2}\left[x^2 - x\right] - G(0)$$

So $u(x,y) = \frac{1}{2}\left[(\operatorname{atan}y - x)^2 - (\operatorname{atan}y - x)\right] + \frac{1}{2}\left[(\operatorname{atan}y + x) - (\operatorname{atan}y + x)^2\right]$

$$\boxed{u(x,y) = x\,(1 - 2\operatorname{atan}y)}$$

Problem 3 [30]

(1) Mathematical formulation

$$\begin{cases} \dfrac{\partial T}{\partial t} = \dfrac{\partial^2 T}{\partial x^2} & \text{(heat equation with } k=1) \\ T(x,0) = \begin{cases} 0 & \text{if } x \in [0,\frac{1}{2}] \\ 1 & \text{if } x \in (\frac{1}{2},1] \end{cases} \\ \dfrac{\partial T}{\partial x}\Big|_{x=0} = \dfrac{\partial T}{\partial x}\Big|_{x=1} = 0 & \text{(insulating boundaries)} \end{cases}$$

(2) As $t \to +\infty$ the solution reaches a steady state which satisfies

$$\begin{cases} \dfrac{\partial^2 T_\infty}{\partial x^2} = 0 \\ \dfrac{\partial T_\infty}{\partial x}\Big|_{x=0} = \dfrac{\partial T_\infty}{\partial x}\Big|_{x=1} = 0 \end{cases}$$

$\Rightarrow$ $T_\infty(x) = ax + b$; $\Rightarrow$ to satisfy the bcs, we must have $a = 0$

$$\Rightarrow \quad T_\infty(x) = b.$$

How do we determine b?

Way #1 Note that $\displaystyle\int_0^1 T(x,t)\,dx$ = constant.

Indeed:
$$\int_0^1 \frac{\partial T}{\partial t}\,dx = \int_0^1 \frac{\partial^2 T}{\partial x^2}\,dx = \left[\frac{\partial T}{\partial x}\right]_0^1 = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\int_0^1 T\,dx = 0 \Rightarrow \int_0^1 T\,dx = \text{constant}$$

So:
$$\int_0^1 T_\infty(x)\,dx = b = \int_0^1 T(x,0)\,dx = \frac{1}{2}$$

So $b = \frac{1}{2}$

Note: Using your "physical intuition to get b is also OK, but then you need to prove it's correct.

(3) Separation of variables: let $T(x,t) = A(x)B(t)$

$$\Rightarrow \begin{cases} \frac{dB}{dt} = \lambda B \\ \frac{d^2A}{dx^2} = \lambda A \end{cases}$$

- we ignore solutions with $\lambda > 0$ since they are unphysical (growing exponentials; x-equation cannot fit bcs)

- for $\lambda = 0 \Rightarrow$ $\left.\begin{array}{l} A_0(x) = ax + b \\ B_0(t) = \text{constant} \end{array}\right\}$ that's the steady-state solution found earlier.

- for $\lambda < 0$ let: $\Lambda^2 = -\lambda$

$$A_n(x) = a_n \cos \Lambda_n x + b_n \sin \Lambda_n x$$

to fit bcs we need $b_n = 0$ and $\Lambda_n = n\pi$ $\quad (n \geq 1)$

$\hookrightarrow$ $A_n(x) = a_n \cos(n\pi x) \qquad (n \geq 1)$

$\hookrightarrow$ $B_n(t) = e^{-n^2\pi^2 t}$

So the solution becomes

$$T(x,t) = b + \sum_{n=1}^{\infty} a_n \cos(n\pi x) e^{-n^2\pi^2 t}$$

To find the $\{a_n\}$ coeffs, we apply the initial conditions

$$b + \sum_{n=1}^{\infty} a_n \cos(n\pi x) = \begin{cases} 1 \text{ if } x \in [0, \frac{1}{2}] \\ 0 \text{ if } x \in (\frac{1}{2}, 1] \end{cases} = f(x)$$

$\uparrow$ if you haden't determined b yet, now you can still do it (way # 2)

$\Rightarrow$ This is a Fourier Series for an even function of period 2 $\Rightarrow$ construct $\tilde{f}$, even, periodic extension of f, then

$$b = \frac{1}{2} \int_{-1}^{1} \tilde{f}(x)\,dx = \int_0^1 f(x)\,dx = \frac{1}{2}$$

$$a_n = 1 \int_{-1}^{1} \tilde{f}(x)\,dx = 2\int_0^1 f(x)\cos(n\pi x)\,dx$$

$$= 2\int_{1/2}^{1} \cos(n\pi x)\,dx = \frac{-2}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

[30] Problem 4 Cantilever Beam $U_{tt} = -\alpha^2 U_{xxxx}$

BCs $u(0,t) = 0$ $U_x(0,t) = 0$
$U_{xx}(1,t) = 0$ $U_{xxx}(1,t) = 0$

ICs $u(x,0) = f(x)$
$u_t(x,0) = g(x)$

(1) Separation of variables let $U = A(x)\,B(t)$

$$\Rightarrow \quad A\ddot{B} = -\alpha^2 B A''''$$

so and $\begin{cases} \ddot{B} = -\alpha^2 \lambda B \\ A'''' = \lambda A \end{cases}$ Note that we expect oscillations in time so $\lambda > 0 \Rightarrow$ let $\lambda = \Lambda^4$

The solutions for $A(x)$ are then $A'''' = +\Lambda^2 A$ so

$$A_n(x) = a_n \cos(\Lambda_n x) + b_n \sin(\Lambda_n x) + c_n \cosh(\Lambda_n x) + d_n \sinh(\Lambda_n x)$$

To satisfy the boundary conditions we need
$A_n(0) = 0$ $A_n'(0) = 0$ $A_n''(1) = 0$ $A_n'''(1) = 0$

$$\Rightarrow \quad A_n'(x) = \Lambda_n \left[-a_n \sin(\Lambda_n x) + b_n \cos(\Lambda_n x) + c_n \sinh(\Lambda_n x) + d_n \cosh(\Lambda_n x) \right]$$

$$A_n''(x) = \Lambda_n^2 \left[-a_n \cos(\Lambda_n x) - b_n \sin(\Lambda_n x) + c_n \cosh(\Lambda_n x) + d_n \sinh(\Lambda_n x) \right]$$

$$A_n'''(x) = \Lambda_n^3 \left[+a_n \sin(\Lambda_n x) - b_n \cos(\Lambda_n x) + c_n \sinh(\Lambda_n x) + d_n \cosh(\Lambda_n x) \right]$$

$$\Rightarrow \begin{cases} a_n + c_n = 0 & \rightarrow a_n = -c_n \\ b_n + d_n = 0 & \rightarrow b_n = -d_n \\ -a_n \cos\Lambda_n - b_n \sin\Lambda_n + c_n \cosh\Lambda_n + d_n \sinh\Lambda_n = 0 \\ a_n \sin\Lambda_n - b_n \cos\Lambda_n + c_n \sinh\Lambda_n + d_n \cosh\Lambda_n = 0 \end{cases}$$

so we have

$$\begin{cases} c_n(\cos\Lambda_n + \cosh\Lambda_n) + d_n(\sin\Lambda_n + \sinh\Lambda_n) = 0 \\ c_n(-\sin\Lambda_n + \sinh\Lambda_n) + d_n(\cos\Lambda_n + \cosh\Lambda_n) = 0 \end{cases}$$

↳ a homogeneous linear system. This only has non-trivial solutions if the determinant is 0, that is if

$$(\cos\Lambda_n + \cosh\Lambda_n)^2 - (\sin\Lambda_n + \sinh\Lambda_n)(-\sin\Lambda_n + \sinh\Lambda_n) = 0$$

$\Rightarrow$

$$\cos^2\Lambda_n + 2\cos\Lambda_n\cosh\Lambda_n + \cosh^2\Lambda_n + \sin^2\Lambda_n - \sinh^2\Lambda_n = 0$$

$$\Rightarrow \quad 2 + 2\cos\Lambda_n\cosh\Lambda_n = 0 .$$

or $\boxed{\cos\Lambda_n\cosh\Lambda_n = -1}$

(2) To solve this numerically, for example, plot the function $\cos x\cosh x + 1$ & look around the intercepts with x-axis. (that's the "quick & dirty way")

→ find
$\Lambda_1 \cong 1.8751$
$\Lambda_2 \simeq 4.69409$
$\Lambda_3 \simeq 7.85476$
$\Lambda_4 \simeq 10.9955$
$\Lambda_5 \simeq 14.1372$

Other methods
→ Newton Raphson

Finally, the eigenfrequencies of oscillation of the beam are given by $\omega_n = \alpha\sqrt{\lambda_n} = \alpha\Lambda_n^2$ so

$\omega_1 \cong 3.516\alpha$
$\omega_2 \cong 22.034\alpha$
$\omega_3 \cong 61.697\alpha$
$\omega_4 \cong 120.78\alpha$
$\omega_5 \cong 199.65\alpha$