Handout 1: Midterm Review Checklist

This worksheet will help you prepare for the Midterm. If you make sure you master all of these points, you will do great!

Studying tips: Read each section. If you think you master the material, briefly look over the sample problems to convince yourself you know how to do them (if you're not sure how to do it, then try, and check your answers in the back,otherwise, quickly move on to the next section). If you don't think you master the material, go through the textbook and lecture notes about the material, make sure you understand the examples. Then do as many sample problems as you can, checking your answer in the back. The bottom line is: don't spend too much time on the points you know - make sure you spend most of your time studying for the points you're less familiar with.

Finally, if you think there is a typo somewhere, please email me! pgaraud@ams.ucsc.edu

1 Algebra Reviews

This section contains a number of things you should know from classes prior to joining this one. If you are not comfortable with the material, make sure to work on it!

1.1 Intervals

See Section 1.1.

- You need to know the correct notations for intervals, in particular the difference between open and closed intervals. Also be sure to know how to write intervals which extend to either $+\infty$ or $-\infty$.
- You need to be able to write intervals as inequalities, and vice-versa

Example:

- $x > 2 \Leftrightarrow x \in (2, +\infty)$
- $x \leq 2 \Leftrightarrow x \in (-\infty, 2]$

Sample problems: page 4-5, problems 41-52

1.2 Integer exponents

See Appendix B1. You need to be completely comfortable manipulating expressions with integer powers. This involves *knowing* the following formulae (i.e. knowing where they come from, and how to use them without mistake). For any expression E, and any integer n or m:

$$E^0 = 1$$
$$E^{n+m} = E^n E^m$$
$$E^{-n} = \frac{1}{E^n}$$
$$E^{n-m} = \frac{E^n}{E^m}$$
$$E^{nm} = (E^n)^m = (E^m)^n$$

So for example:

- $100^0 = 1$, $y^0 = 1$, $(2x+1)^0 = 1$, $\left(\frac{1}{x^2-3x-2}\right)^0 = 1$
- $2^2 2^3 = 2^{2+3} = 2^5$, $x^2 x^8 = x^{2+8} = x^{10}$, $(y^2-1)^4 = (y^2-1)(y^2-1)^3$.
- $100^{-2} = \frac{1}{100^2}$, $\frac{1}{x^4} = x^{-4}$, $(4x^2-1)^{-1} = \frac{1}{4x^2-1}$.
- $\frac{2^{15}}{2^2} = 2^{13}$, $\frac{(x^2+1)^3}{(x^2+1)^2} = (x^2+1)^{3-2} = (x^2+1)$, $\frac{(x^2+1)^3}{(x^2+1)^{-2}} = (x^2+1)^{3-(-2)} = (x^2+1)^5$
- $2^6 = (2^3)^2 = 8^2$, $16^2 = (4^2)^2 = 4^4$, $((x-1)^2)^{-2} = (x-1)^{2\times(-2)} = (x-1)^{-4}$

You need to be able to simplify expressions with integer exponents using these formulae. See Appendix B1 for many examples/practise problems.

Note: don't fall into the standard traps:

- $(a^n)^m$ IS NOT a^{n+m}.
- $(a+b)^n$ IS NOT $a^n + b^n$

Sample problems: page A-13 (back of book), problems 19-38

1.3 Factoring

See Appendix B4 (and 2.1 and 2.2). You need to know how to factor an expression with respect to a particular variable. For this purpose, you should think about this approach to the problem:

- Is the expression one of the standard formulae for factoring?
  1. Is it a difference of squares $a^2 - b^2$? In that case factor as $(a-b)(a+b)$
  2. Is is something like $a^2 + 2ab + b^2$? In that case factor as $(a+b)^2$
  3. Is is something like $a^2 - 2ab + b^2$? In that case factor as $(a-b)^2$
- Is it a quadratic expression of the kind $ax^2 + bx + c$? If that's the case, see the Section on Quadratics.
- If it's not a standard expression, or a quadratic, is there an obvious common factor? If so, begin by factoring it out, and then deal with the next expressions using the same chart.
- If you can't see a common factor, can you group terms in pairs (or sometimes triplets) which can each be factored ? If that's the case, try that, and see if the remaining factors then become a "common factor"

This about covers all the possibilities you will encounter in this class. All it takes now is practise. See Appendix B4 and Sections 2.1 and 2.2 for lots of practise. You should know the standard formulae for factoring *by heart*.

Sample problems: page A-30, problems 1, 11 (use the proper quadratic formula), 19, 21, 39, 55, 59.

1.4 Fractions

See Appendix B5. You need to be completely comfortable manipulating expressions with fractions. This involves being able to reduce expression to the same denominator, and simplify compound fractions without making any errors.

Reducing to same denominator. Remember that you CANNOT write $\frac{a}{b} + \frac{c}{d} = \frac{a+b}{c+d}$. First you have to find the common denominator for the two fractions. The simplest way is to multiply the first fraction by $\frac{d}{d}$ (which is 1) and the second fraction by $\frac{b}{b}$ (which is also1). Then

$$\frac{a}{b} + \frac{c}{d} = \frac{a}{b}\frac{d}{d} + \frac{c}{d}\frac{b}{b} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad+bc}{bd}$$

This is true for any expression a, b, c or d. Examples:

$$\frac{2}{x+3}+\frac{3x}{4-x}=\frac{2}{x+3}\frac{4-x}{4-x}+\frac{3x}{4-x}\frac{x+3}{x+3}=\frac{2(4-x)+3x(x+3)}{(x+3)(4-x)}$$

Sometimes, other common denominators are simpler, as in the example

$$\frac{x}{(x+1)}+\frac{2x}{(x^2-1)}$$

The expression x^2-1 can be rewritten $(x-1)(x+1)$ and so contains an $x+1$ already. So, to reduce to a common denominator, we write

$$\frac{x}{(x+1)}+\frac{2x}{(x^2-1)}=\frac{x}{(x+1)}\frac{x-1}{x-1}+\frac{2x}{(x^2-1)}=\frac{x(x-1)+2x}{(x-1)(x+1)}$$

Simplifying compound fractions: Compound fractions usually are ratios of fractions or expressions containing fractions. The rule there is

- Step 1: Identify the main fraction "bar". Reduce the expression on top of that to the same denominator, and the expression below that to the same denominator. Example:

$$\frac{\frac{1}{x}+\frac{2}{x+1}}{5-\frac{1}{2x+1}}=\frac{\frac{1}{x}\frac{x+1}{x+1}+\frac{2}{x+1}\frac{x}{x}}{5\frac{2x+1}{2x+1}-\frac{1}{2x+1}}=\frac{\frac{(x+1)+2x}{x(x+1)}}{\frac{5(2x+1)-1}{2x+1}}=\frac{\frac{3x+1}{x(x+1)}}{\frac{10x+4}{2x+1}}$$

- Step 2: use the rule that

$$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\frac{d}{c}$$

(i.e. flip the denominator up and multiply with the numerator). In the example above it becomes

$$\frac{\frac{3x+1}{x(x+1)}}{\frac{10x+4}{2x+1}}=\frac{3x+1}{x(x+1)}\frac{2x+1}{10x+4}=\frac{(3x+1)(2x+1)}{x(x+1)(10x+4)}$$

- Simplify the resulting fraction if possible. In the example above there is no way of simplifying further, but sometimes we may see expression which can be simplified such as

$$\frac{x(x-1)}{2x-2}=\frac{x(x-1)}{2(x-1)}=\frac{x}{2}$$

or

$$\frac{x^2-2}{\sqrt{2}-x}=\frac{(x-\sqrt{2})(x+\sqrt{2})}{\sqrt{2}-x}=-(x+\sqrt{2})$$

since for any expression E, $E/(-E)=-1$, in particular

$$\frac{x-y}{y-x}=-1$$

For practise with these expression and simplification, see Appendix B5.

Finally, remember that under no circumstances are you allowed to simplify fractions which are not factored: for example

$$\frac{x^2+(x+2)}{a+(x+2)}\neq\frac{x^2}{a}$$

So, whenever simplifying rational expressions, always factor the numerator and denominator first! (see factoring section)

Sample problems: page A-35 and A-36: problems 15, 19, 23, 33, 39, 41

2 Functions

2.1 Definition and basic use of a function

See Section 3.1, 3.2 and 3.5

- You need to know how to identify that a rule is, or isn't a function
  Sample problems: page 141, problems 4-8
- You need to know how to determine that a graph is, or isn't the graph of a function (i.e. vertical line test)
  Sample problems: page 153, problems 5,6
- You need to know that every point on the graph of a function has coordinates $(x, f(x))$
  Sample problems: page 153-154, problems 1-4, 15,16
- You need to be able to determine the domain of definition of a given function
  Sample problems: page 141, problems 9-16
- You need to be able to evaluate functions at any given point (e.g. $f(2), f(x^2), f(x+h), f(a)$)
  Sample problems: page 142, problems 29-38
- You need to know the basic operations on functions (sum, product, quotient, ...)
  Sample problems: page 189, problems 1-6
- You need to know how to compose functions $f \circ g(x) = f[g(x)]$ and $g \circ f = g[f(x)]$
  Sample problems: page 190, problems 10-16

2.2 Graphs of basic functions

Graphs of basic functions: You need to know how to *sketch* the following functions without using a calculator. Sketching means "to draw something quickly, without too much attention to detail or perfect accuracy, but nevertheless represent any salient feature correctly".

- $f(x) =$ a constant (e.g. $f(x) = 2$, $f(x) = -\pi$)
- $f(x) = ax + b$
- $f(x) = |x|$
- $f(x) = x^2$, $f(x) = x^3$, $f(x) = x^4$, $f(x) = x^5$, etc...
- $f(x) = \sqrt{x}$
- $f(x) = \frac{1}{x}$, $f(x) = \frac{1}{x^2}$, $f(x) = \frac{1}{x^3}$, $f(x) = \frac{1}{x^4}$, etc...

See Section 3.4

Graphs of functions which are translated vertically and horizontally: Given the graph of a function $f(x)$, you have to know how to graph the functions

- $f(x) + a$: the graph is moved up or down (if a is positive or negative respectively) by an amount a
- $f(x + a)$: the graph is moved left or right (if a is positive or negative respectively) by an amount a

Graphs of functions which are reflected across the $x-$axis and $y-$axis: Given the graph of a function $f(x)$, you have to know how to graph the functions

- $-f(x)$: the graph is reflected across the $x-$axis
- $f(-x)$: the graph is reflected across the $y-$axis

Sample problems: page 179, 3-40

2.3 Inverse of functions

See Section 3.6.

- You need to know how to determine graphically that a function has an inverse (i.e. horizontal line test)
- You need to know the notation: the inverse of $f(x)$ is the function denoted by $f^{-1}(x)$.
- You need to know that $f^{-1}(x)$ DOES NOT mean $\frac{1}{f(x)}$.
- You need to be able to calculate the inverse of simple functions (i.e. solve the equation $y = f(x)$ for x.
  Example: $f(x) = \frac{2x-1}{3x+1}$. To calculate the inverse, solve $y = \frac{2x-1}{3x+1}$. The answer is $x = \frac{y+1}{2-3y}$. Then the inverse function is $f^{-1}(y) = \frac{y+1}{2-3y}$ or in other words $f^{-1}(x) = \frac{x+1}{2-3x}$.
- You need to know that the inverse f^{-1} applied to f yields x, and same for f applied to f^{-1}: $f[f^{-1}(x)] = f^{-1}[f(x)] = x$. You need to know how to use this fact to verify that the inverse you calculated is correct:

$$f[f^{-1}(x)] = f\left[\frac{x+1}{2-3x}\right] = \frac{2\frac{x+1}{2-3x}-1}{3\frac{x+1}{2-3x}+1} = \frac{\frac{2x+2-(2-3x)}{2-3x}}{\frac{3x+3+(2-3x)}{2-3x}} = \frac{2x+2-2+3x}{2-3x}\frac{2-3x}{3x+3+2-3x} = \frac{5x}{5} = x$$

- You need to know and understand the relationship between the graph of a function and the graph its inverse (i.e. they are mirror images with respect to the $y = x$ line). You need to be able to use that knowledge to find the graph of f^{-1} based on the graph of $f(x)$.

Sample problems: pages 203-204 problems 3, 4, 9-22

2.4 Linear functions

Equations of lines/graphs of linear functions See Section 1.6. You need to know, and know how to use the equation of a line, in particular

- how to calculate the slope of a line going through two points A(x_A, y_A) and B(x_B, y_B)
  $s = \frac{y_B - y_A}{x_B - x_A}$
- the point-slope formula
  $y - y_A = s(x - x_A)$ for the line with slope s going through the point A(x_A, y_A).
- the slope-intercept formula
  $y = sx + b$ for a line with slope s and $y-$intercept b
- the fact that if two lines are parallel they have the same slopes
- the fact that if two lines are perpendicular the product of their slopes is -1
- how to verify that a point is on the line

Sample problems: page 54-55, problems 15-36

2.5 Quadratic functions

See Section 4.2. For a given function $f(x) = ax^2 + bx + c$ you need to know

- how to find the $y-$intercept (e.g. the $y-$intercept is c)
- how to complete the square to transform $f(x)$ into the vertex form $f(x) = a(x - x_V)^2 + y_V$.
  $f(x) = a\left(x^2 + \frac{b}{a}x\right) + c = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c = a\left(x + \frac{b}{2a}\right)^2 - a\frac{b^2}{4x^2} + c$
- You need to know and understand from basic transformations of the $y = x^2$ parabola *why* $\left(x_V = -\frac{b}{2a}, y_V = f(x_V)\right)$ are the coordinates of the vertex of the parabola.
- based on the sign of a, whether the parabola opens up ($a > 0$) or down ($a < 0$)
- how to find $x-$intercepts (e.g. how to solve the equation $ax^2 + bx + c = 0$) using the discriminant method, see Handout 1.
- how to factor the quadratic depending on the discriminant (see below)
- how to draw a signs table for the quadratic, and how the signs table differs in the case $D = b^2 - 4ac$ is positive, zero and negative.
- how the signs table relates to the graph of the function f.

Discriminant method for quadratic equations: (see Section 2.2) The solutions to the equation $ax^2 + bx + c =$ depends on the value of D:

$$D = b^2 - 4ac$$

- if $D < 0$ there are no solutions. The expression $ax^2 + bx + c$ cannot be factored.
- if $D = 0$ there is one solution $x_1 = -\frac{b}{2a}$. The expression $ax^2 + bx + c$ can be factored as $a(x - x_1)^2$
- if $D > 0$ there are two solutions.

$$x_1 = \frac{-b - \sqrt{D}}{2a} \text{ , } x_2 = \frac{-b + \sqrt{D}}{2a}$$

  The exression $ax^2 + bx + c$ is factored as $a(x - x_1)(x - x_2)$

Examples:

- $x^2 + 6x + 10 = 0$: $D = 6^2 - 4 \times 10 \times 1 = 36 - 40 = -4$. In that case there are no solutions, and the quadratic $x^2 + 6x + 10$ cannot be factored
- $-2x^2 + 12x - 18 = 0$: $D = 12^2 - 4 \times (-18) \times (-2) = 144 - 144 = 0$. In that case there is one solution,

$$x = -\frac{12}{2 \times (-2)} = \frac{12}{4} = 3$$

  and the factored form of the expression is $-2(x - 3)^2$.
- $x^2 - 6x + 4 = 9$: $D = 6^2 - 4 \times 4 \times 1 = 36 - 16 = 20$. In that case there are two solutions,

$$x = \frac{-(-6) + \sqrt{20}}{2 \times (1)} = \frac{6 + 2\sqrt{5}}{2} = 3 + \sqrt{5}, x = \frac{-(-6) - \sqrt{20}}{2 \times (1)} = \frac{6 - 2\sqrt{5}}{2} = 3 - \sqrt{5}$$

  and the factored form of the expression is $\left(x - (3 + \sqrt{5})\right)\left(x - (3 - \sqrt{5})\right) = \left(x - 3 - \sqrt{5}\right)\left(x - 3 + \sqrt{5}\right)$

Sample problems: page 89, problems 43-50 (and factor the quadratics)

2.6 Higher order polynomials

See section 4.6. You need to know

- How to expand a polynomial, and recognize the leading-order term (the $a_n x^n$ term).
- That the behavior of the graph of the polynomial for very large x is the same as the behavior of the leading order term. How to deduce from that whether $f(x)$ goes to $+\infty$ or $-\infty$ as x goes to $+\infty$ or $-\infty$.
- How to recognize that a polynomial is fully factored or not
- How to factor it, if it is not fully factored (see Handout 1)
- How to draw a signs table for the polynomial
- How to deduce from the signs table what the shape of the graph is.
- How to find the behavior near a root.

Sample problems: pages 299-300 problems 27-44 (including, find the behavior near $+\infty$ and $-\infty$, and draw a signs table.)