

## 2.3 Higher-order polynomials

*Textbook section 4.6*

### 2.3.1 Definition and examples

DEFINITION:

VOCABULARY:

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EXAMPLES:

In order to understand the behavior of these polynomials, let's start with studying the behavior of individual terms: functions which are simple powers of  $x$ .

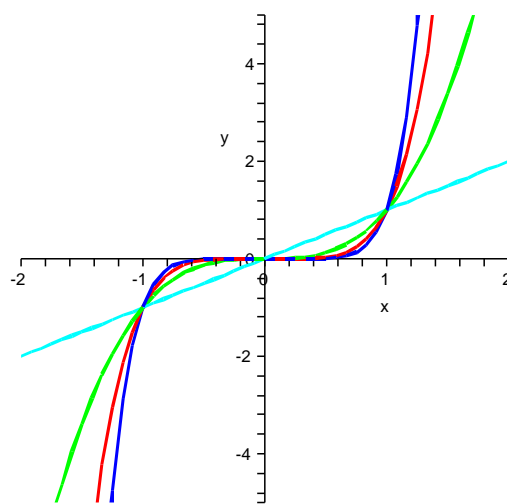
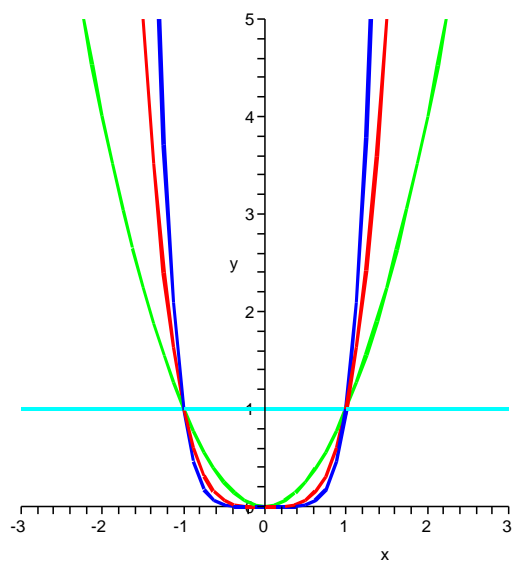
### 2.3.2 Functions of the kind $f(x) = x^n$ with $n$ a natural number

REAL-LIFE  $x^n$  FUNCTIONS: Power functions with integer powers arise naturally in geometrical problems. For example:

- The circumference of a square as a function of side length:
- The area of a square as a function of side length:
- The volume of a cube as a function of side length:
- The surface area of a cube as a function of side length:
- The circumference of a circle as a function of radius:
- The area of a circle as a function of radius:
- The volume of a sphere as a function of radius:
- The surface area of a sphere as a function of radius:

NOTE: Because we live in a three-dimensional world, a volume is always related to the cube of the meaningful lengthscale of the problem, and an area is always related to the square of the meaningful lengthscale. If we lived in a 4-D world ...

The shape of the graphs of functions of the kind  $f(x) = x^n$  depends on whether  $n$  is an even or an odd number.



NOTE:

### 2.3.3 Functions of the kind $f(x) = ax^n$ with $n$ a natural number

When the power is multiplied by a number  $a$ , note that

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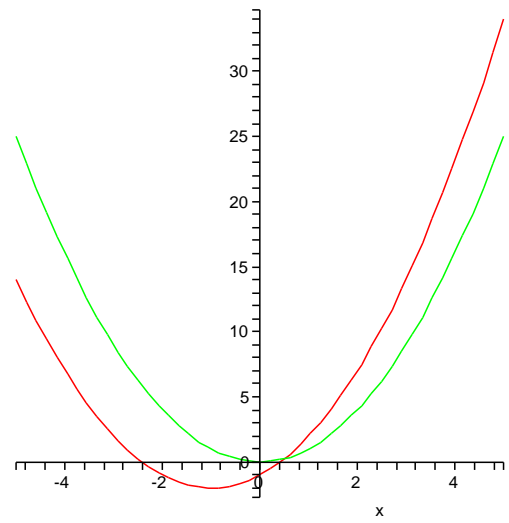
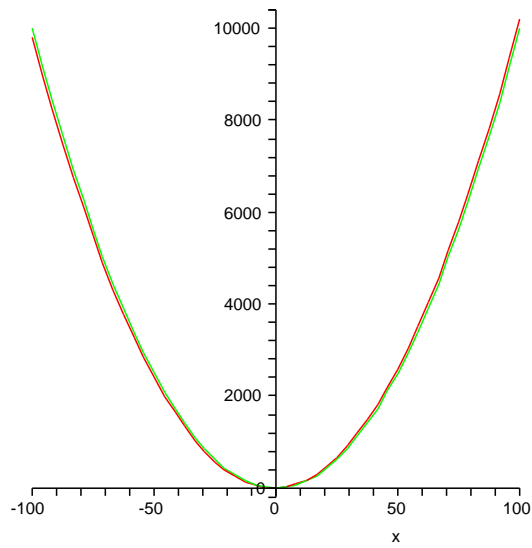
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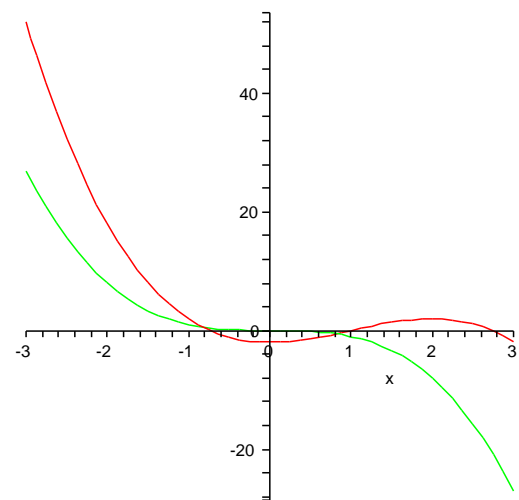
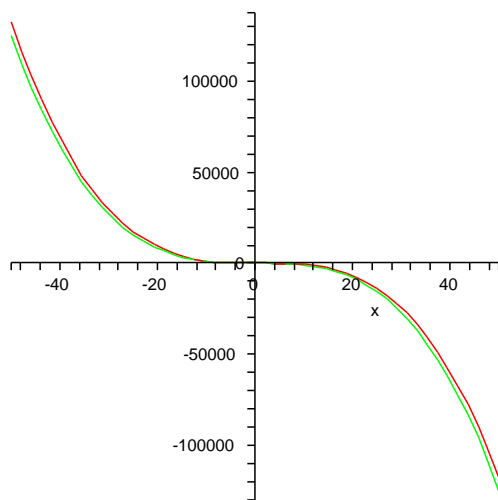
#### 2.3.4 Approximations of polynomials for very large values of $|x|$

The overall shape of the graph of a polynomial function, when  $x$  is either very large and positive ( $x$  going to  $+\infty$ ) or very large and negative ( $x$  going to  $-\infty$ ), resembles that of the corresponding power function of the leading order term.

EXAMPLE 1:  $f(x) = x^2 + 2x - 1$



EXAMPLE 2:  $f(x) = -x^3 + 3x^2 - 2$



CONCLUSION:

### 2.3.5 Factored polynomials

In Chapter 1, we learned how to factor polynomial expressions. We now revisit the problem, and learn what the factored form of a polynomial function tells us about the graph of the function.

**Formal definition of factored form:**

EXAMPLES:

- $f(x) = -(2 + x)(x + 3)^3$

- $f(x) = (x - 1)(2 - x^2)$

- $f(x) = -2x(x^2 - 2x + 1)(x + 3)$

- $f(x) = x^3 + 2x^2 + 4x$

### 2.3.6 Signs tables

Signs Tables are an excellent tool to determine the *sign* of any polynomial function, quadratic or higher-order. Knowing the sign of a function is often a very useful tool for graphing, and for finding out the domain of definition of a function.

IMPORTANT NOTE: Signs tables can only be used if the function is already broken down into its factors.

HOW TO DRAW A SIGNS TABLE:

- Draw the table
- Write **all** the factors vertically on the left
- Write **all** the roots horizontally on the top (in the correct order)
- Draw vertical lines below each root
- Determine and write the sign of each factor; write zeros where appropriate.
- Multiply the signs in each interval to determine the sign of the function.

EXAMPLES OF QUADRATIC FUNCTIONS:

EXAMPLE: Draw a signs table and sketch the function  $f(x) = 4(x - 1)(x + 2)$ .

EXAMPLE: Draw a signs table and sketch the function  $f(x) = -2(1 - x)^2$ .

EXAMPLE: What is the domain of definition of  $f(x) = \sqrt{2x^2 + 2x - 1}$ ?

HIGHER-ORDER POLYNOMIAL FUNCTIONS:

To create a signs table for a fully factored higher-order polynomial, simply follow the same method as for quadratic functions:

EXAMPLES:

- In which interval(s) is the function  $f(x) = -(2 + x)(x + 3)^3$  positive?

- Sketch  $f(x) = (x - 1)(2 - x^2)$ .

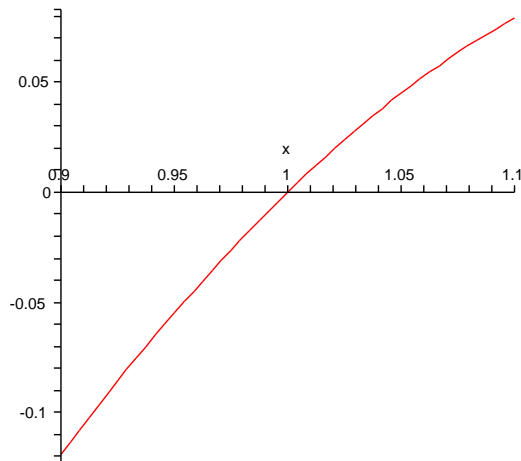
- Sketch  $f(x) = -2x(x^2 - 2x + 1)(x + 3)$

- Find the domain of definition of  $f(x) = \sqrt{x^3 + 2x^2 + 4x}$ .

### Behavior near a root

While the signs table typically gives you most of the information you need about the function, we saw that in some cases the situation is not so clear-cut. In these cases, it is also useful to study the behavior of the function *in the vicinity* of a root, to double-check the signs table and sometimes to find out more about the function.

EXAMPLE 1: Consider the function  $f(x) = (x - 1)(2 - x^2)$ . Near  $x = 1$ , of course,  $f(x)$  is close to 0 (because the  $x - 1$  term becomes very small). But what does it look like?





Since  $x$  is close to 1, let's see what happens if we plug  $x = 1$  into *all the factors except*  $(x - 1)$ . Then we get

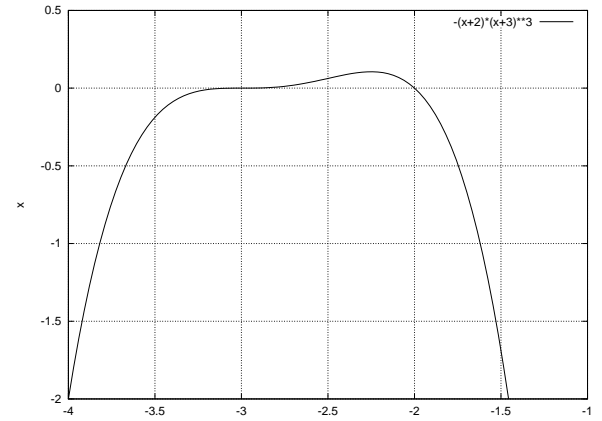
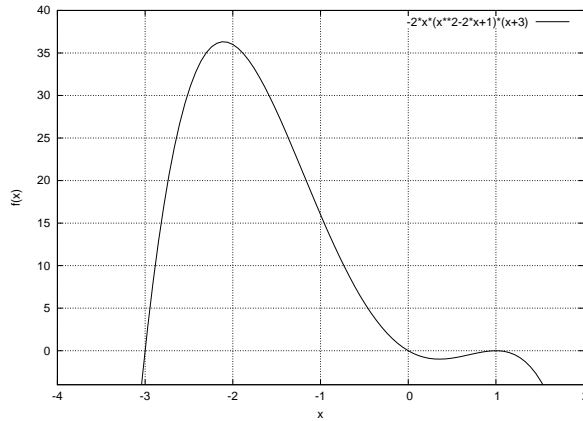
When plotted on the same plot, the two functions are indeed very close to each other for  $x$  near 1.

NOTE: This can also help us check the signs table: the table says that  $f(x)$  goes from negative to positive as  $x$  goes through 1. The line  $y = x - 1$  also goes from negative to positive as  $x$  goes through 1.

EXAMPLE 2: Consider the function  $f(x) = -2x(x^2 - 2x + 1)(x + 3)$ . What does it look like near  $x = 1$ ?

EXAMPLE 3: Consider the function  $f(x) = -(2 + x)(x + 3)^3$ . What does it look like near  $x = -3$ ?

We can now check with a graphing device that our guesses are indeed correct:



### 2.3.7 Example of application of polynomials

The creators of Ice Age want to produce another short involving Scrat (the prehistoric squirrel) in which he falls through an ice-slide while chasing his acorn. They want the profile of the slide to look somewhat like the one one depicted on the figure below. In order to animate the short, they need to program that profile in their software in the form of a function (height as a function of distance). What kind of function would you suggest they use?