2.2 Quadratic functions

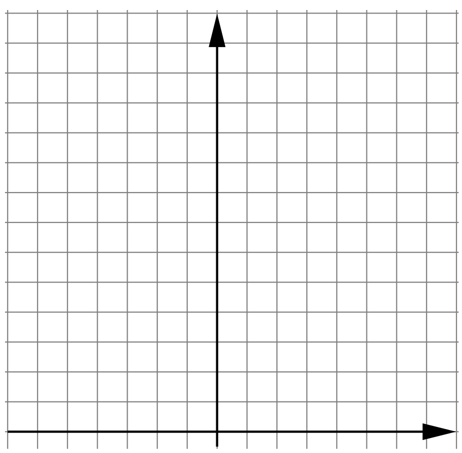
Textbook Section 2.2 and 4.2

2.2.1 Definition and basic $f(x) = x^2$ function

The general expression for a quadratic function is

Some quadratic functions (but not all of them!) can be factored, so that:

The simplest example of a quadratic function is the function $f(x) = x^2$:



In fact, the graph of all quadratic functions is a *parabola*. The exact shape and position of the parabola depends on the coefficients of the quadratic. Different cases can arise:

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2.2.2 Behavior as $x \to \pm\infty$

Whether a parabola opens "up" or "down" can very easily be determined simply by inspection of the quadratic function. The following mini-inquiry will help you find this out for yourself.

Let's consider two examples of quadratic functions:

- $f(x) = 3x^2 - 2x - 100$
- $g(x) = -2x^2 + x + 10$

Complete the following tables, and interpret your findings with your partner.

x	$3x^2$	$-2x$	100	$f(x) = 3x^2 - 2x - 100$
-1000				
-100				
-10				
-1				
0				
1				
10				
100				
1000				

x	$-2x^2$	x	10	$g(x) = -2x^2 + x + 10$
-1000				
-100				
-10				
-1				
0				
1				
10				
100				
1000				

We notice that:

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Conclusion:

2.2.3 Behavior as $x \to 0$

What the parabola looks like near the $y-$ axis (i.e. when x is close to 0) can very easily be determined simply by inspection of the quadratic function. The following mini-inquiry will help you find this out for yourself.

Let's consider two examples of quadratic functions:

- $f(x) = 3x^2 - 2x - 1$
- $g(x) = -x^2 + x + 2$

Complete the following tables, and interpret your findings with your partner.

x	$3x^2$	$-2x$	-1	$f(x) = 3x^2 - 2x - 1$	$-2x - 1$
-1					
-0.1					
-0.01					
0					
0.01					
0.1					
1					

x	$-x^2$	x	2	$g(x) = -x^2 + x + 2$	$x + 2$
-1					
-0.1					
-0.01					
0					
0.01					
0.1					
1					

We notice that:

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Conclusion:

2.2.4 Why does a parabola cross the $x-$axis?

Whether a parabola crosses the $x-$ axis or not depends on whether the corresponding quadratic function can be factored or not. This will be the subject of the next few sections of this chapter. This mini-inquiry will help you understand why this is the case.

Consider the quadratic functions $f(x) = x^2 + 4$ and $g(x) = x^2 - 4$.

- Sketch these functions (with a pencil) using what you know about translation of functions

- What do you notice about the graphs and the $x-$axis?

- Factor $g(x)$

- Under which conditions is $g(x) = 0$?

- Re-draw your sketch of $g(x)$ more precisely using that information.
- Can $f(x)$ be factored? If yes, what is the factored expression? If no, why can it not be factored?

Now consider the quadratic functions $f(x) = (x + 2)^2 + 2$ and $g(x) = (x - 2)^2 - 1$.

- Sketch these functions (with a pencil) using what you know about translation of functions

- What do you notice about the graphs and the $x-$axis?

- Based on the results from the previous part of this inquiry, do you think $f(x)$ and $g(x)$ can be factored? In each case, if yes, try to find the factored form. If no say why. Redraw your original sketches if needed using that information.

Conclusion:

2.2.5 REVIEW: Quadratic formulas and completing the square

In order to become proficient at factoring quadratics (and higher-order polynomials) there are 3 formulas that must *absolutely* be known by heart:

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These can be sometimes be used directly to factor quadratics as in the examples below:

EXAMPLES:

- $f(x) = x^2 - 9$
- $f(x) = -x^2 + 1$
- $f(x) = 2x^2 - 3$
- $f(x) = x^2 - 6x + 9$
- $f(x) = -2x^2 - 4\sqrt{3}x - 6$

Noticing expressions which use these formula can be tricky! You are strongly encouraged to do the "Algebra review sheet 2" for more practice.

However, in some cases the quadratic expression we have doesn't quite fit the formula. In that case, we can *always* try to **complete the square** first to gain more information about the quadratic, and perhaps factor it further...

"Completing the square" involves matching the first two terms in a quadratic expression to one of the two formulas above (depending on the signs in the expression), as in the examples below.

EXAMPLE 1: $f(x) = x^2 - 2x + 3$.

EXAMPLE 2: $f(x) = 3x^2 - 3x - \frac{1}{4}$.

EXAMPLE 3: $f(x) = -\frac{1}{2}x^2 - 3x + 1$.

Note how in all cases, the "completed formula" can be used to find out about the graph of the quadratic functions, using the graphical transformation properties studied in Chapter 1.

IMPORTANT RESULT:

2.2.6 Application: optimization problems

In Homework 2, we studied a problem in which we had to find the area of a pasture enclosed by a 500ft long fence on 3 sides, and a river on the third side. The area enclosed can be expressed as a function of the length of the side of the rectangle perpendicular to the river as follows:

This quadratic expression can be manipulated by completing the square to get *at the same time* the value of x which maximizes A as well as the value of A at the maximum:

2.2.7 From a completed square to a factored expression

The general expression of a completed square looks like

We can see that there are 4 scenarios, based on the signs:

Hence the cases with opposite signs can be factored, and the ones with the same signs cannot. This can in fact be seen mathematically as well if we remember the expression

so that

This method can therefore be used in a very systematic way to find the factored form of a quadratic, if it exists, or to know for sure that it cannot be factored. Let's go back to our earlier examples:

Example 1: $f(x) = x^2 - 2x + 3$.

Example 2: $f(x) = 3x^2 - 3x - \frac{1}{4}$.

Example 3: $f(x) = -\frac{1}{2}x^2 - 3x + 1$.