1.9 REVIEW: Solving equations for a variable

Textbook Chapter 1.3

1.9.1 The basics

In many real-life problems, we are seeking a numerical answer to a specific question. *How many people need to go to see Toy Story 3 for the movie to break even? For the movie to gross $100M?* Solving these problems can often be done by casting them into a mathematical framework. The typical approach is the following:

- Give the unknown number you're seeking a name. x is fairly typical of course, but any letter, symbol or name can do.
- Relate that unknown variable to other known quantities of the problem using all the information available. This creates an equation for the unknown. Note that sometimes, more than one unknown may be needed. In that case, you will need as many equations as unknowns to solve the problem.
- Solve the equation for the unknown. This usually involves simple (or more complex) algebraic manipulations to transform the original equation, step by step, into something which expresses the unknown explicitly in terms of everything else, such as

$$x = ...$$

Examples: How many people need to go to see Toy Story 3 for the movie to break even? For the movie to gross $100M?

These examples were quite easy because the equation that needed to be solved was a simple linear equation. Let us now work on more complicated examples.

1.9.2 What to do with more complicated equations?

There are two main steps towards solving more complicated equations.

- Simplify both sides of the equation as much as possible
- "Move" all terms containing the unknown in one side of the equation, and continue to reduce/simplify until only the unknown is left.

1. Simplifying step: This may involve expanding an expression (see Algebra Worksheet 1), or perhaps factoring it (see next Review), seeing which terms may cancel. Simplifying a sum or different of terms

involving fractions often involves reducing them to the same denominator first, then simplifying the numerator and denominator separately.
Examples:

- $3[1 - 2(x + 1)] = 2 - x$

- $\frac{1}{x+5} = \frac{2}{x-3} + \frac{2x+2}{(x+5)(x-3)}$

2. Shuffling step: To "move" an expression (i.e. a number, variable, a set of terms, etc) from one side of the equation to the other, always remember that this corresponds to a specific algebraic equation – either adding/substracting, or multiplying/dividing depending on the situation.

- In an expression of the kind $A = B+C$, "moving C to the left-hand-side" is equivalent to substracting C on both sides. So when C is moved over, it becomes $-C$:

- In an expression of the kind $A = B - C$, "moving C to the left-hand-side" is equivalent to adding C on both sides. So when C is moved over, it becomes $+C$:

- In an expression of the kind $A = \frac{B}{C}$, "moving C to the left-hand-side" is equivalent to multiplying C on both sides. So when C is moved over, it moves *up*:

- In an expression of the kind $A = \frac{C}{B}$, "moving C to the left-hand-side" is equivalent to dividing C on both sides. So when C is moved over, it moves *down*:

NOTE: Be careful! The best way to move a whole expression around is to put it in brackets, and only remove the brackets once it's on the other side. Otherwise, it's very easy to make a sign error or a multiplication/division error.

Examples:

- $-3 - 6x = 2 - x$

- $\frac{1}{x+5} = \frac{4x+12}{(x+5)(x-3)}$

- $4 = \frac{2x+2}{x-1}$

- $(x-1)(x+2) = (x+1)(x-2)$

1.9.3 The domain of the variable and forbidden solutions

Some of the equations presented above have an additional subtelty which does not occur often, but which you should nevertheless be aware of.

The original equation is not necessarily defined for all values of the variable. For example,

- $4 = \frac{2x+2}{x-1}$

- $\frac{1}{x+5} = \frac{2}{x-3} + \frac{2x+2}{(x+5)(x-3)}$

In the first case, we found that the solution was , which is in the domain of the variable - no problem. In the second case, however, the solution was , which is *not* in the domain of the variable. This solution is therefore NOT ALLOWED.

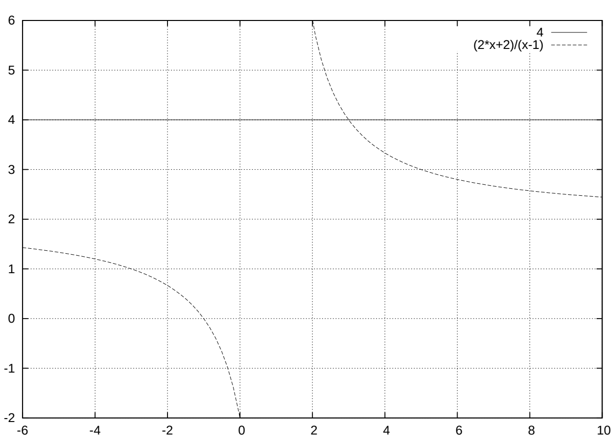
1.9.4 Note on nonlinear equations

Note how all the equations in the previous sections eventually reduced to a linear equation. Of course, this is not a general situation! In this class, we will learn to solve a number of equations which are not linear. But bear in mind that there are in fact many equations that do not have an analytical solution (i.e. a solution where you can write the answer as a formula). In some cases, the only way to find a solution to an equation is to do it graphically or numerically (using special software).

The graphical method is particularly easy to use, and will also tell you immediately how many solutions to expect. To do it for an equation of the kind $f(x) = g(x)$, where f and g are two functions of x, simply

- Plot the graphs $y = f(x)$ and $y = g(x)$ on the same plot.
- The $x-$coordinate of the intersection(s) of the two graphs mark the solution(s) of the equation.

Example: The solution to the equation $4 = \frac{2x+2}{x-1}$ can be found graphically on this plot!



1.10 The inverse of a function

Textbook section 3.6

1.10.1 Definition and examples

Definition:

As a result

Graphically:

- $y = f(x) = 3x + 2$:

- $y = f(x) = x^2$ (for $x \geq 0$):

- $y = f(x) = \sqrt{x-2}$ (for $x \geq 2$):

IMPORTANT NOTES:

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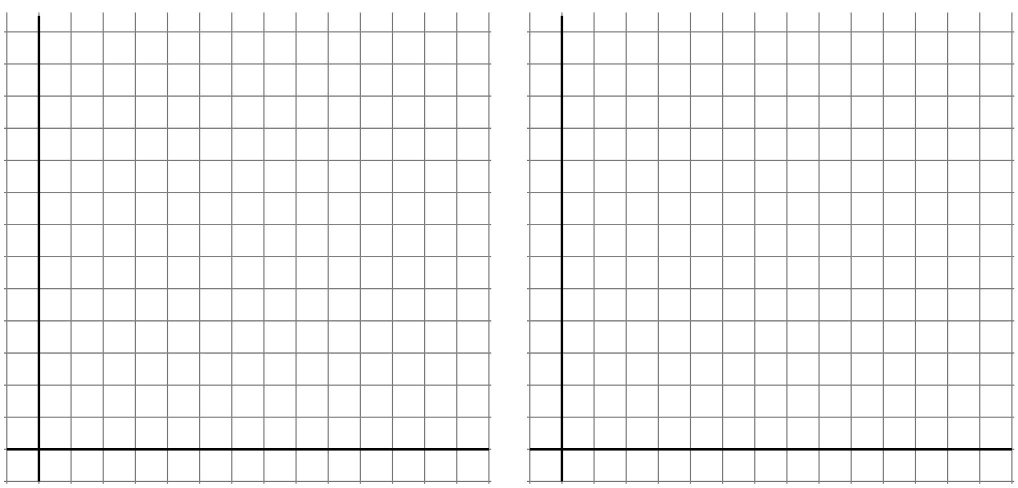
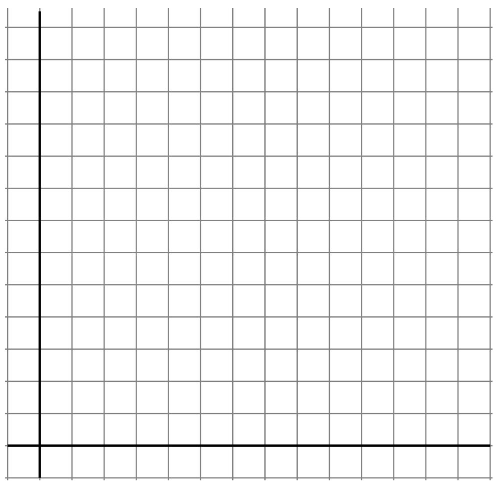
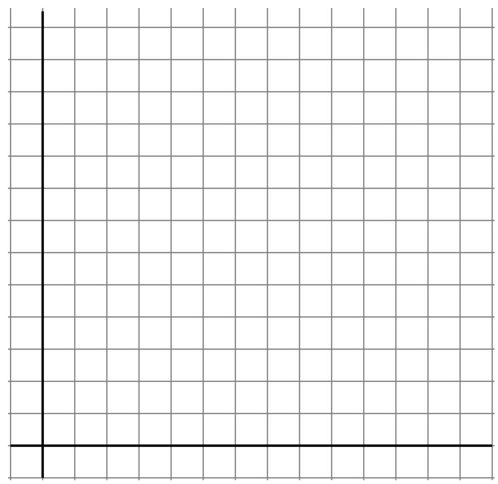
1.10.2 Graph of an inverse function and horizontal line test:

EXAMPLE 1: $y = f(x) = 3x + 2$



EXAMPLE 2: $y = f(x) = x^2$





So from these graphs we notice that:

NOTE: It may happen that through this process, the graph of the inverse does not satisfy the vertical line test: in that case, the inverse is not uniquely defined.

HORIZONTAL LINE TEST: To verify that the inverse of a function is unique, we check that the function satisfies the horizontal line test:

When a function $f(x)$ does not satisfy the horizontal line test, we can often choose a smaller domain for which the inverse *is* unique.

EXAMPLE: for the function $f(x) = x^2$, we saw earlier that the inverse of $f(x) = x^2$ is defined provided we select only the interval for which $x \geq 0$. In this interval, the function $f(x)$ does satisfy the horizontal line test.

1.10.3 More examples of inverse functions