In this Lecture we will start applying the principles we saw in the very first class, on how to approach a scientfic problem. Here, we will focus on the following steps:

- Create a mathematical model
- Analyse your model critically

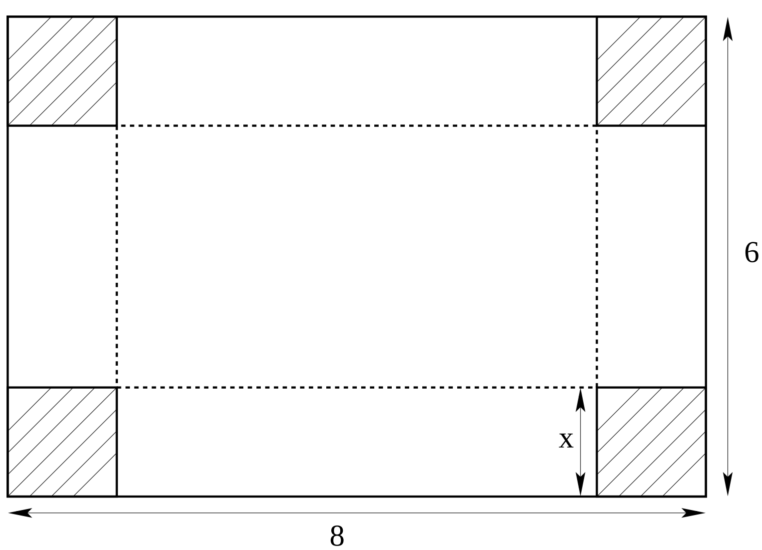
1.5 How to construct a mathematical function

Textbook Section 4.4

In this section we will study a number of problems which lead to the construction of a mathematical function to relate two quantities. The usual question will be "What is the mathematical function which describes the relationship between quantity X and quantity Y". As such, the answer will no longer be a single number, as you may be used to from Algebra or previous mathematical classes, but must instead be an expression, which says, $Y = f(X)$ where f is the function to be determined. The textbook contains a large number of examples from geometry. We will also see a few examples from Economics, and from biology.

1.5.1 Textbook geometrical example

This example is from the Textbook homework number 45, page 269. Given a cardboard box of size 6×8 inches, we cut out the four corners and fold it along the dotted line to make a box. If we cut out squares of size $x \times x$ inches, what is the volume f of the box we can thus make as a function of x?



1.5.2 Example from economics

The movie Toy Story 3 production costs were \$75M, with an additional \$25M marketing costs. The movie tickets for the 3D version were sold at an average of \$15/person, and those for the 2D version were sold at an average \$10/person across all cinemas in the US. Assuming that half of the people who went to see the movie saw the 3D version, and the other half saw the 2D version, what is the gross G of the movie as a function of the percentage P of all Americans who went to see it? (You may assume for simplicity that the population of the US is 300 million).

1.5.3 Example from Biology

A petri dish contains an initial population of 20 bacteria. These reproduce by cellular division every 2 hours (meaning that 1 bacteria splits into 2 bacteria every 2 hours). What is the number of bacteria B in the petri dish as a function of the number of hours H since the beginning of the experiment?

1.6 How to use available software for graphing the functions

Now that we know the functions, we can plot them on a graph. The graphs will give us important insight on the problem. Graphing these functions can be done in a number of different ways:

- By creating a table of data containing pairs of points $(x, f(x))$ and plotting them by hand
- Using a graphing calculator
- Using a graphing package on a computer (e.g. Excel, gnuplot, etc...)

In what follows, we will explore the two last possibilities. Since graphing data and functions is something you will often have to do in your career, it is important that you familiarize yourself with the available tools as soon as possible.

Before we start, however, it is important to bear one point in mind. Mathematical functions are often defined for a range of values of the independent variable much larger than what you may actually be interested in plotting. To save time, it is important to think, ahead of time, what is the *useful* range of values of the independent variable we wish to illustrate.
Example:

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Graphing functions with a graphing calculator. All graphing calculators have in-built graphing capabilities. If you have a graphing calculator, it is your responsibility to learn how to use it to plot graphs. Make sure you know how to:

- Plot the graph of a function
- Select the range of $x-$ and y-values to best illustrate the graph (as described above)
- If necessary, select logarithmic axes (see later in the course)

Graphing with Excel: To graph a function with Excel, you need

- *Create a set of values of x.* For this purpose, first choose the lowest value of x you are interested in plotting, and write it in a cell. In the cell just below, write the next number in the list of values you want to plot. If you highlight the two cells with the mouse, and then move the mouse to the lower right corner of the second cell, you will see a narrow black + sign appear. Click on the + sign and drag it down the colum, and you will see a series of numbers appear in each subsequent cell, each incremented from the previous one by the difference of the first 2 numbers written at the beginning. *For example*: if you start with 0, 1, the subsequent numbers are 2, 3, 4, 5, ... If you start with 63, 63.2, then the next numbers will be 63.4, 63.6, 63.8, etc..
- *Create the function.* To do this, you first do it for one single cell. For example, in the cell horizontally adjacent to your first $x-$value, write the sign =, and then the formula you want to use. Each time x would appear in the formula, replace it by clicking on the cell *containing* the first value of x. *For example*: if I want to create the function $f(x) = x(x - 1)$, and if my first $x-$value is in the cell A2, then in the cell B2 I type "= A2*(A2-1)".

- *Create the list of $y = f(x)$ values.* To complete the table and create all other y-values, we simply need to apply the rule to the rest of the $x-$values. To do this, right-click on the original formula (e.g., the cell B2 of the previous example), and copy it (Ctrl-C in Windows, Cmd-C in Mac). Then left click on the same cell, and now drag the mouse down (holding the click) through the whole column, down to the cell adjacent to your last value of x. Now you can paste the formula in all cells (Ctrl-V in Windows, Cmd-C in Mac). Voila!

- *Graph the function.* To graph the function, go to the "Gallery" button, or the "Charts" button (depending on the version of excel you use. Select the two columns you have just created, then click either on the line plot, or the scatter plot. This should create your graph in a separate window.

- *Re-select axes if necessary.* Once the plot has been created, you can improve the selection of the axes by clicking on them. This opens up a friendly and fairly intuive window giving you many options.

NOTE: While graphing with Excel is quite cumbersome, it does have the advantage of giving you the table of x and y values, while other softwares typically don't. Also, most majors in PBSci will require you to use Excel at some point, so you may as well learn to use it now!

Graphing with Gnuplot: On Macs (and Linux-based systems), a very convenient program exists called Gnuplot. Install gnuplot if you can. To graph a function with Gnuplot, simply

- Launch gnuplot by typing "gnuplot" in an X11 or Xterm window

- Type "plot [your formula with x]", as for example, "plot $x * (x - 1)$". This will create the plot with default axes.

- If you want to center the graph differently, you need to specify axes: for example, to force the x-range of values to be between 0 and 1, and the $y-$range of values to be between -1 and -0.5, type: "plot [0:1] [-1:-0.5] $x * (x - 1)$". Voila!

1.7 How to analyse your results

Once the function has been plotted, one can begin the analysis. Ask yourself the following questions:

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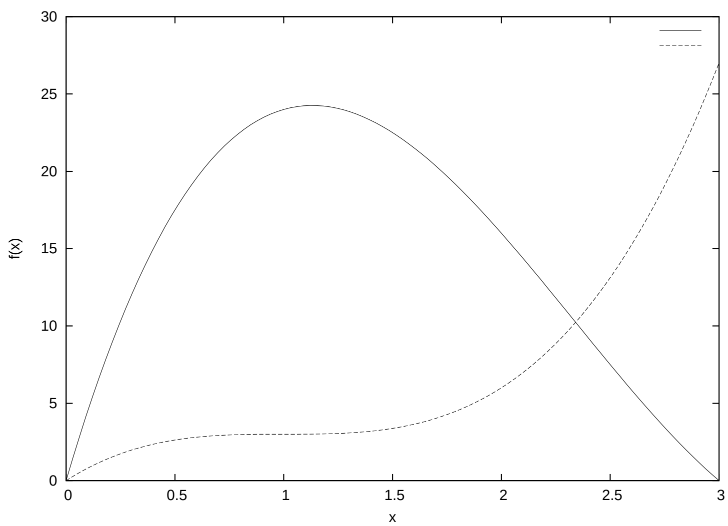
Examples: (Pairwork)

- Look at the plots on the next page. In each case, one is correct and the other one isn't. Can you decide why just by thinking of the real problem at hand?

- Once you have identified the correct plot, what can you learn from the graph?



30
25
20
15
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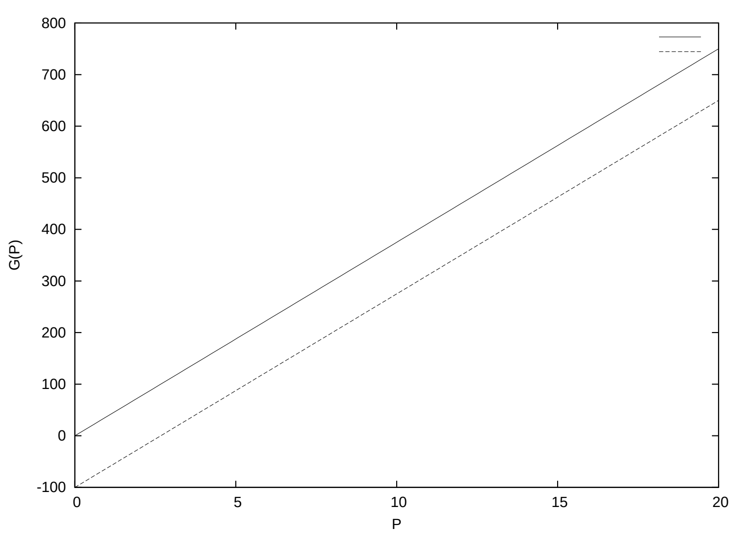
f(x)
0
0.5
1
1.5
2
2.5
3
x



800
700
600
500
400
300
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-100
G(P)
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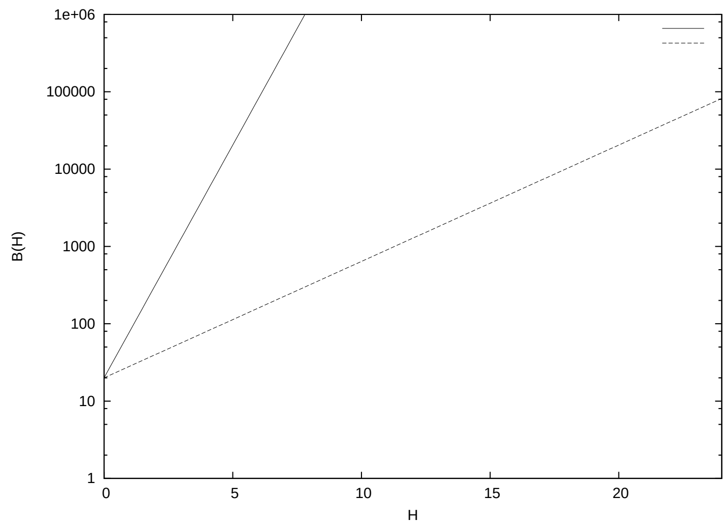
5
10
15
20
P



1.8 Operations with functions

1.8.1 Standard algebraic operations

Functions can be manipulated in the same way as expressions: you can add or substract two functions, multiply or divide them (provided you're not dividing by zero). We saw some examples of this in the previous section.
Examples:

- Sum of two functions:

- Difference of two functions:

- Product of two functions:

- Ratio of two functions:

However, be very careful that the following statements are not true in general:

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1.8.2 Composite functions

Physical quantities can depend on each other through a series of causal links: a quantity A depends on a quantity B, and B depends on C, so indirectly, A also depends on C. We also saw an example of this in the previous section.

Examples:

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To represent this mathematically, we use an *operator* on functions called *composition*.

Examples:

- $f(x) = \sin(x)$, $g(x) = 4x - 1$: $f \circ g$:

- $f(x) = \frac{1}{x^2-2}$, $g(x) = x+1$: $f \circ g$:

- $f(x) = \sqrt{1-x}$, $g(x) = x^2$: $g \circ f$:

NOTE: Changing the order of the composition yields an entirely different function!

Example: $f(x) = \sqrt{x}$, $g(x) = x^2 + 1$

- $f \circ g$:

- $g \circ f$: