Chapter 5

Trigonometric functions

In this final Chapter we learn about the basic trigonometric functions, including the sine function and the cosine function, and others such as tangent, co-tangent, secant and co-secant. Note that we will mostly skip Chapter 6 in the textbook, except for 6.1 and 6.5.

5.1 Degrees and radians

Textbook Section 7.1

There are two major ways of measuring angles in geometry: in *degrees* and in *radians*.

The degree measure was introduced historically in astronomy to measure the displacements of stars, and is based on the fact that there are approximately 360 days in a year (well, there are in fact 365.25 days in a year, but 360 conveniently divides nicely by 2, 3, 4, 6, 10, 12, .., while 365.25 doesn't).

The radian measure is the one more commonly used in mathematics. It is based on the length of portions of a circle:

Based on this we have the correspondance:

To summarize, to go between radians and degrees and vice-versa,

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By convention, in mathematics we also define a direction to an angle:

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Since the circle wraps around, an angle is always defined up to a value of 2π:

5.2 Right-angle triangles and basic trigonometric functions

Textbook Section 6.1

5.2.1 Sine, cosine and tangent

Sine, cosine and tangent functions are usually defined through their association with right-angle triangles:

IMPORTANT CONSEQUENCES: From this diagram, we see that there are two very important formulae relating these three basic trigonometric functions to one another:

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5.2.2 Co-tangent, secant and cosecant

There are three more important functions to learn, defined as follows:

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These 6 functions altogether form the basic trigonometric functions you will need to know.

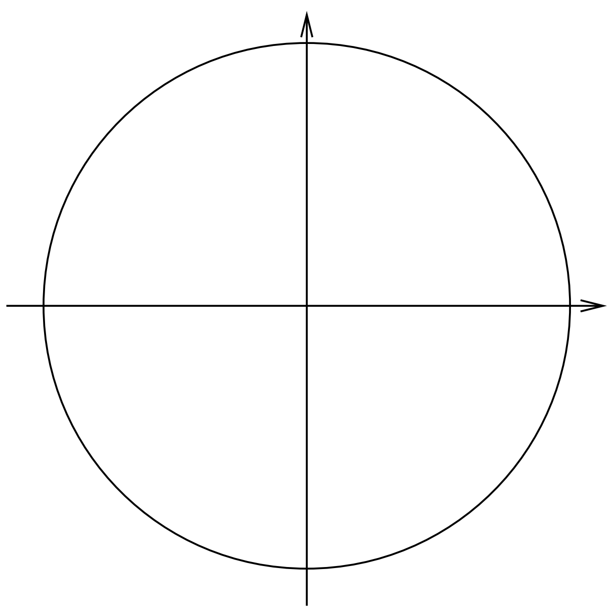
5.3 The unit circle, and the graphs of sine, cosine and tangent

Textbook Section 7.4

5.3.1 Construction of the unit circle

The unit circle is a wonderfully convenient way of *visualizing* the sine and cosine functions.

DEFINITION:



Based on this, we can already deduce some particular values of the sine, cosine and tangent functions:

5.3.2 Sine and Cosine of important angles

In addition to $\pi/2$, π, $3\pi/2$ and 2π, there are 3 important angles for which you need to know the sine and cosine of:

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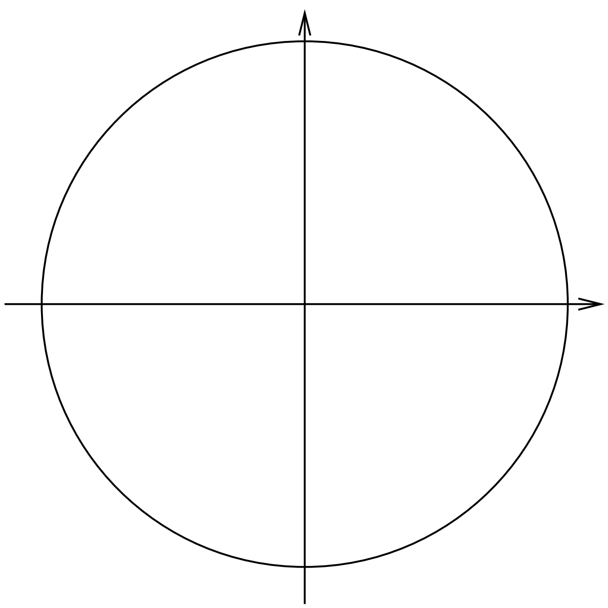
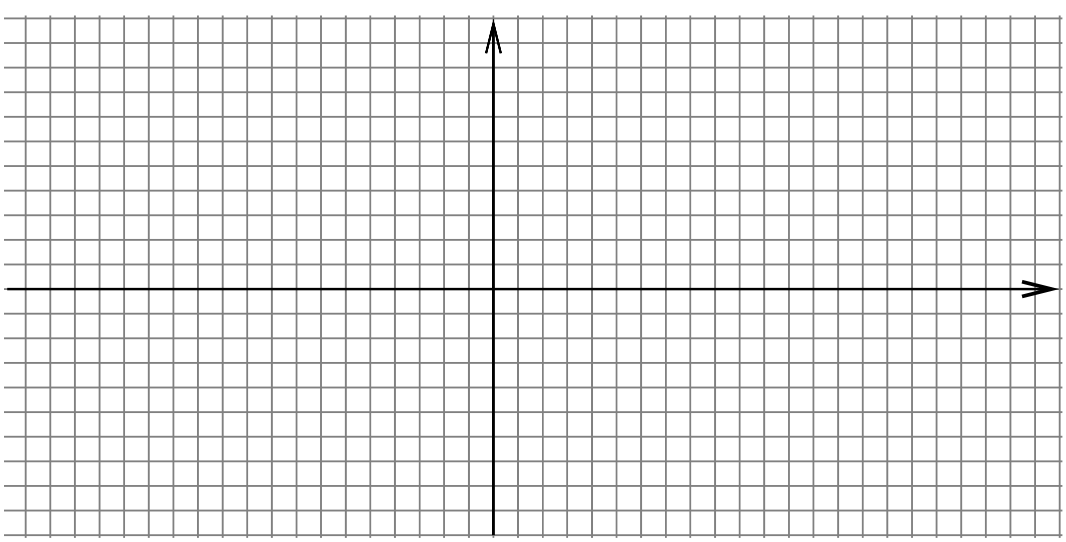
Based on the unit circle, we can now find the sine and cosine of many other angles:



Finally, we can use this information to plot the sine and cosine functions:



5.3.3 What can we deduce from the graphs of $\sin(x)$ and $\cos(x)$?

Based on the graphs of $\sin(x)$ and $\cos(x)$, we see that

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5.3.4 Periodicity

DEFINITION:

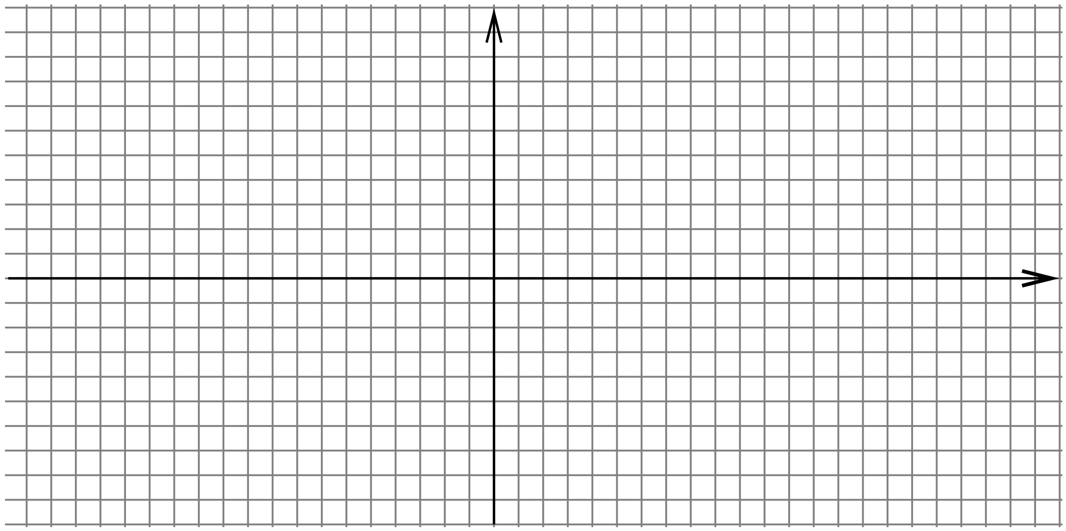
Based on the unit circle construction, and the fact

we conclude that

5.3.5 The graph of the tangent function

Textbook Section 7.7

The graph of the tangent function can be deduced from the graphs of the sine and cosine functions between $-\pi$ and π, and the fact that all basic trigonometric functions are 2π-periodic.



5.4 Oscillatory functions in general

Textbook Section 7.5

5.4.1 Examples of oscillations in nature

Many real phenomena are prone to *oscillations*. An oscillation is defined a very regular periodic behavior, and can usually be expressed in terms of sine and cosine functions. Natural examples of oscillations in nature are:

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Oscillations are usually characterized by 4 numbers:

- The mean:

- The amplitude of oscillation:

- The period of oscillation:

- The phase of oscillation:

5.4.2 Modeling oscillations

To model these various possible changes in oscillatory behavior, we can modify the basic trigonometric functions. Based on how the properties of the graphs of functions are changed, we see that

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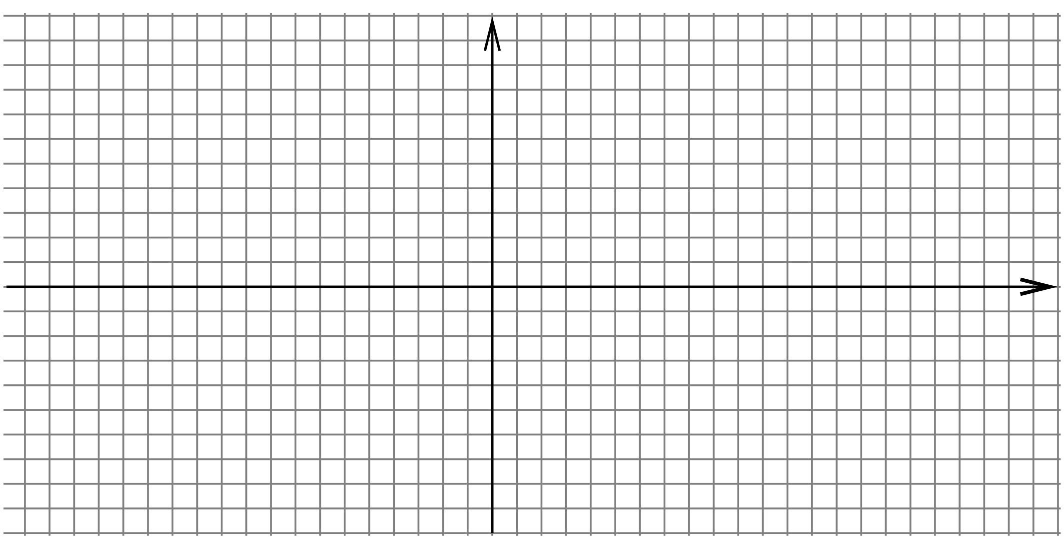
But how do we change the period? To answer this question, let's begin by graphing the function $f(x) = 2x$:



NOTE: It appears that the period of the function $\cos(2x)$ is π instead of 2π. Can we prove this?

5.4.3 General properties of the functions $f(x) = m + a\cos(cx + d)$ and $g(x) = m + a\sin(cx + d)$

As we saw in the previous sections, the functions $f(x) = m + a\cos(cx + d)$ and $g(x) = m + a\sin(cx + d)$ are oscillatory functions with the following properties:

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Note that the phase is often defined in different ways depending on applications, and also because sine and cosine themselves are actually the same function with a different phase (see previous sections).

Finally, the period of the oscillation can be deduced by asking the question: for which value of p is $f(x + p) = f(x)$ (and same for g). To find the period, we therefore have to solve the equation

This shows that the only number which changes the period of a function is the number multiplying x in the argument of sin or cos. More generally, we have just shown that

Example: What is the mean, period and amplitude of the following functions:

- $f(x) = 2 + 2\cos(2x + 2)$
- $f(x) = 3 + e\sin(\pi x)$
- $f(t) = \sin(2\pi t - 1)$

5.5 Trigonometric formulas

There are a few very important formulas in trigonometry, which you will need to know as a preparation for Calculus. These formulas are very useful when trying to simplify complicated expressions which involve trigonometric functions.

5.5.1 The basic formulas

Textbook Section 6.5

We've already seen a few of these formulas: the three basic ones are

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Examples of use: Simplify:

- $1 + \tan^2(x)$

- $1 - \frac{\cot^2(x)}{\csc^2(x)}$

The list of trigonometric identities which can be proved using the basic formulas is endless. See Textbook page 478 for a few of them.

5.5.2 The addition formulas

Textbook Section 8.1 and 8.3

The addition formulas relate the sines and cosines of *sums* of angles to the *products* of sines and cosines of basic angles. The 4 formulas are

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Examples: These examples show that these formulas indeed work:

- $\cos(\pi/3 - \pi/6)$

- $\cos(\pi/3 + \pi/3)$

- $\sin(\pi/4 + \pi/4)$

Also, they confirm our hunch that $\sin(x)$ and $\cos(x)$ are the "same" function but displaced by $\pi/2$. Indeed,

But where do these formulas come from? Proving the last one will be done in Section, and the others can be deduced from it...

Note: There is no equivalent *product* formulas: there is no simple identity for $\cos(ab)$ and $\sin(ab)$. On the other hand, the addition formulas can be used backward to prove the following identities:

Alltogether these identities can be used to prove another nearly-infinite number of identities... for example

$$\frac{\sin(a-b)}{\cos(a)\cos(b)} + \frac{\sin(b-c)}{\cos(b)\cos(c)} + \frac{\sin(c-a)}{\cos(c)\cos(a)} = 0$$

5.5.3 The double-angle formulas

Textbook Section 8.2

As a consequence of the addition formulas, we have 3 more formulas which are called the *double-angle* formulas because they express the sine and cosine of the angle $2a$ in terms of the sine and cosine of the angle a. These formulas are

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To show them, note that

Again, these double-angle formula can be used to simplify trigonometric expressions: for example

$$\frac{\sin(2x)}{\sin(x)} - \frac{\cos(2x)}{\cos(x)} = \sec x$$

Indeed

5.5.4 Summary

Of the various formulas above, only a few have to be known by heart:

- The definition of the various functions:

- The Pytagorean identity:

- The symmetry property:

- The periodicity property:

- The phase-shift property:

- The 2 double-angle formulas:

All of the other formulas can be recovered from combining these ones – so don't clog your memory with formulas... In what follows, we will now use these formulas to solve equations involving trigonometric functions.

5.6 Solving trigonometric equations

Textbook Section 8.4

The formulas studied earlier will now be used to first simplify, and then actually solve trigonometric equations. Let's work by examples.

EXAMPLE 1: Solve the equation $\sin(2x) = \cos(x)$

This example illustrates two steps: using the standard formulas to simplify an equation, and using the unit circle to find how many solutions there are. Another way of doing the second step is to graph the simplified equation:

EXAMPLE 2: Solve the equation $\sin(3x) = \frac{\sqrt{3}}{2}$

This example illustrates that you must be very careful in *not* forgetting the extra solutions coming from the $+2k\pi$ part of the original solution.

Example 3: Solve the equation $\cos^2(x) + \cos(x) - 2 = 0$

Example 4: Solve the equation $|\sin(x)| = |\cos(x)|$

This equation illustrates the use of the unit circle to solve trigonometric equations and visulalize the solutions...

Example 5: However, not all trigonometric equations can easily be solved analytically using these methods. As a last example, let's investigate:

$$3\sin(x) = 1$$

This illustrates the need to introduce inverse trignometric functions

5.7 The inverse trigonometric functions

Textbook Section 8.5

DEFINITIONS: We define the three basic inverse trigonometric functions:

HORIZONTAL LINE TEST? However, note how neither of the three basic trigonometric functions pass the horizontal line test. As a consequence, the domain of definition of the inverse functions is limited to a region where the function does.

For the $\sin(x)$ function and its inverse:

For the $\cos(x)$ function and its inverse:

For the $\tan(x)$ function and its inverse:

As for any function and their inverse, $f(f^{-1}(x)) = f^{-1}(f(x)) = x$ so we have:

What is interesting, however, is to combine the Pythagorean theorem with the inverse equations to get $\sin(\cos^{-1}(x))$ and $\cos(\sin^{-1}(x))$:

These will be used a lot in Calculus!