4.3 General logarithmic functions

Textbook Section 5.3

4.3.1 Definition and graph

Definition:

Graph:

Case 1: $a > 1$

Case 2: $0 < a < 1$

Domain of definition:

Universal property of logarithms:

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4.3.2 Examples of logarithms in common bases

The function $f(x) = \log_2(x)$ (logarithm in base 2)

The function $f(x) = \log_{10}(x)$ (logarithm in base 10)

Examples:

- $\log_{10}(1000) =$
- $\log_2(0.25) =$

4.3.3 The inverse relationships

Since the logarithm in base a is the inverse of the exponential in base a, we have the two fundamental relationships

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These relationships can be used to simplify expressions with exponentials and logs...
EXAMPLES:

- $\log_2(2^x) =$

- $\log_5(5\sqrt{5}) =$

- $\log_{10}(100^x) =$

- $3^{\log_3(2)} =$

- $10^{\log_{100}(2x)} =$

These relationships can also be used to prove important properties of logarithms...

4.3.4 Properties of the logarithms and examples of use

Textbook Section 5.4

The following rules apply for logarithms.

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To show why these formulas are true, we go back to the definition of the logarithm as an inverse, and use the properties of the exponentials: for instance, to show why the first formula is true write:

The other rules are shown in Section.

EXAMPLES:

- Combine into one log expression: $\log_2(x^2 - 1) - \log_2(x + 1)$

- Simplify $\log_2(8(x - 2))$:

- Simplify $\log_{10}(100^{x+1}) + \log_{10}\left(\frac{1}{5^x}\right)$

4.4 The natural exponential and the natural logarithm

Textbook Section 5.2

4.4.1 Definition

There is one particular base called the *natural* base for exponentials and logarithm.

DEFINITION:

Remember that e is a real number, with value approximately equal to:

Naturally, various function can be constructed from e^x:

Definition:

Properties of the natural logarithm and exponential: since these two functions are inverse of each other...

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The reason why this peculiar base is important in mathematics will be explored in more detail in Calculus. However, for the moment, just accept the following property of the natural exponential:

4.5 Change of base and applications

Textbook Section 5.4

There are a few manipulations of exponential and logarithms which involve changing bases with the natural base. It is important to master these changes of base formulae.

4.5.1 Changing from base a to the natural exponential

The rule of change of base from a base a exponential to the natural exponential (and vice versa) is:

The reason why this works is simple:

Examples:

- $2^x =$
- $\left(\frac{1}{4}\right)^x =$

4.5.2 Application of the exponential change of base

Textbook Section 5.7

This change of base is very often used: since the natural base is *the* standard base, most systems which grow or decay exponentially are expressed in base e rather than another base. For example, we saw that for radioactive decay, the amount of radioactive material decays with time as

Another way of writing this exponential decay using a base e exponential is:

In fact, in most textbook you will find that the decay function is given as where the number r is called the *decay rate*. By comparing the two expressions, note that the half-life τ and the decay rate r are related by

Note that our textbook uses the equivalent formula e^{kt} (where our r is $-k$ in the book).

EXAMPLE 1: Plutonium-241 has a half-life of 13 years. Express the formula for the amount of Plutonium-241 left after t years both as an exponential in base $1/2$, and an exponential in base e.

Given an initial sample of pure Plutonium-241, what percentage will be left after 100 years?

EXAMPLE 2: Radium-226 is another radioactive element. It's amount in any object decays with time following this function:

$$A(t) = A_0 e^{-0.000427t}$$

where t is expressed in years and A_0 is the original amount. What is the half-life of this element?

As another example, we studied the exponential growth of a rabbit population, and assumed that this population doubled every month:

$$R(t) = 2 \times 2^t$$

where t is expressed in (number of) months since the appearance of the first pair of rabbits. After how many months does the number of rabbits exceed the human population on Earth (assume it is 6 billion).

This example also serves to illustrate two more things we need to know how to do:

- Changing from logarithm in base a to natural logarithm
- Solving equations with logarithms and exponentials

4.5.3 Changing from base a to the natural logarithm

The rule of change of base from a base a logarithm to the natural logarithm (and vice versa) is (see result above for example):

The reason why this works is simple too:

EXAMPLES:

- $\log_2(x) =$
- $\log_{\frac{1}{4}} x =$

NOTE: This formula, when applied to a, yields the obvious relationship

If you are not sure of your change-of-base formula, this is a good way of double-checking that the formula you remember is the correct one.

This change of base is particularly useful because most calculators only provide $\ln(x)$ and not $\log_a(x)$. So, whenever you have to calculate $\log_a(x)$, simply evaluate:

Example: We already saw some examples earlier, but here are some more...

- Solve the equation $2^x = 6$

- Show that for any a and b, the following is true: $\log_a(b)\log_b(a) = 1$.

4.6 Solving equations with logarithms and exponentials

Let's now study some additional equations which involve logarithm and exponentials, and yet can be solved analytically.

- Solve the equation $3^{x-1} = 2$

- Find the $x-$ and $y-$intercepts of the function $f(x) = 3^{x+2} - 4$

- Solve the equation $10^{x^2-2x} = 3^x$

- Solve the equation $\log_{10}(4^{x+1}) = \log_2(40)$

- Solve the equation $\ln(\ln(x)) = 2$

- Solve the equation $4^x + 2^x - 1 = 0$