Chapter 4

Powers, exponentials and logarithms

4.1 Power functions

4.1.1 Definition and examples

Definition:

Examples:

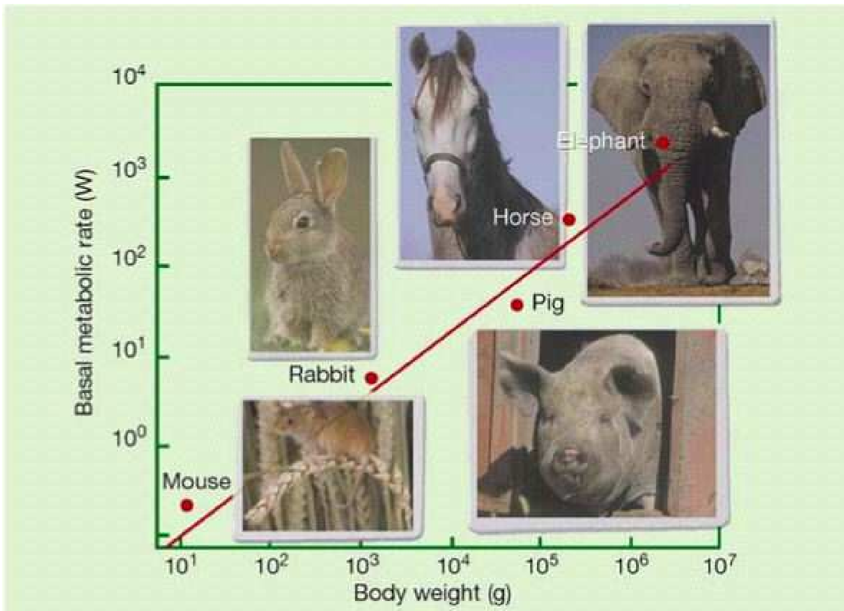
4.1.2 Examples of power functions in Nature

We saw earlier that many geometric problems lead to power-law relationships between variables. However, even in Nature there are many examples of power laws, sometimes with integer exponents, sometimes with non-integer exponents.

- Fundamental forces of Physics follow power-law scalings:
  Example: The force of gravitation

- Allometric laws in Biology: Power functions are frequently found when relating (empirically) two biological variables.

  - EXAMPLE: Kleiber's Laws:
    Body Metabolism Rate = $3.5(\text{Body Weight})^{3/4}$Watts .



  - The optimal cruising speed for a bird/plane as a function of their body mass:
    Speed = $30\text{Mass}^{1/6}$m/s (mass in kg)

4.1.3 Manipulations of power functions

The following rules apply for manipulating power functions:

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EXAMPLES:

4.1.4 The inverse of power functions

Rule:

Proof 1:

Proof 2: Remember that to show that two functions (f and g for example) are inverse of one another, you simply evaluate $f \circ g$:

Aplications:

- What is the inverse of $f(x) = x^3$?
- What is the inverse of $f(x) = x^{-1/2}$?
- What is the inverse of $f(x) = x^{-\pi}$?
- Solve for x: $x^{2\pi} - 5x^{\pi} + 6 = 0$

- Find the inverse of $f(x) = \left(\frac{(2x^2)^{3/7}}{x^{1/7}}\right)^{-14}$

4.1.5 Graphs of power functions

Graph on linear paper: The overall shape of the graph of a power function depends on the sign and value of the exponent...

Graph on log-log paper: (see Section)

The graph of a power-law function on log-log paper is always a straight line! We will see in the next section why this is the case. In scientific papers, when researchers want to illustrate that their data supports a power law relationship between two variables, they plot the data on a log-log plot (see the Allometric relationship plot earlier).

4.2 General Exponential functions

Textbook Section 5.1

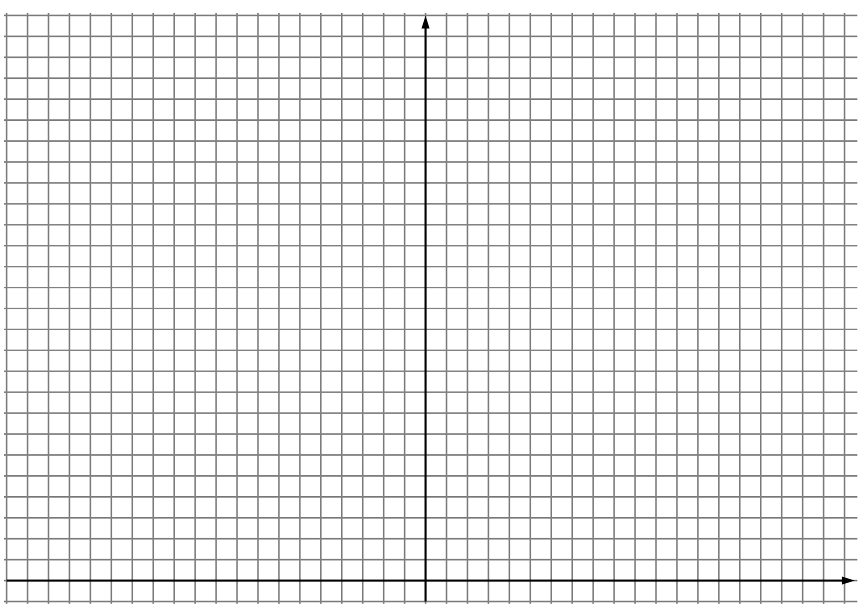
4.2.1 Definition of an exponential functions

DEFINITION:

NOTE: Do not mix up power and exponential functions!

- For power functions:
- For exponential functions:

While we may not be used to thinking of exponents as non-integers, or non-rational numbers, think of the following construction for the function $f(x) = 2^x$:



MANIPULATION OF EXPONENTIAL FUNCTIONS: The rules for manipulating these functions are quite similar to the rules for manipulating powers. Given an exponential function in base a

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Also, given another exponential function in base b

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etc...

EXAMPLES:

- Simplify: $f(x) = \frac{3^{x+2}}{9}$
- Simplify $f(x) = \frac{2^{2x}}{4^x}$
- Simplify $f(x) = 25^x 5^{-x-1}$
- Simplify $f(x) = 2^{2x} 3^x$

4.2.2 Graphs of exponential functions

The graph of an exponential function $f(x) = a^x$ depends on the value of the base a.

Case 1: $a > 1$
Typical example: $f(x) = 2^x$

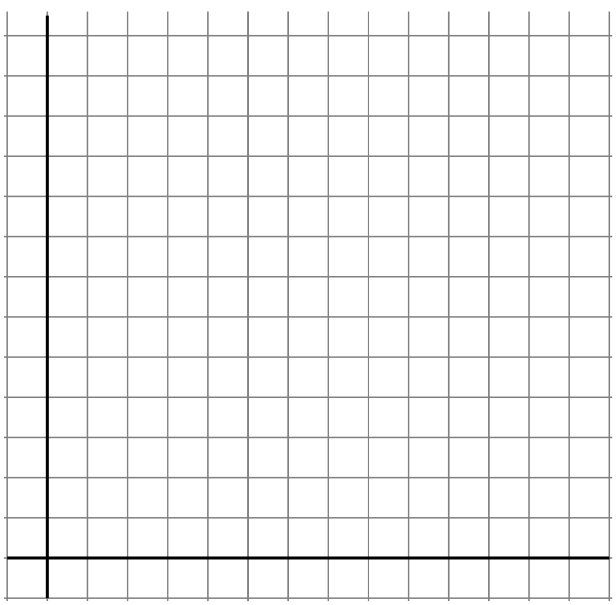
Case 2: $0 < a < 1$
Typical example: $f(x) = \left(\frac{1}{2}\right)^x$

4.2.3 Applications of exponential functions

Exponential functions occur in Nature very commonly in systems where something doubles/triples/quadruples... in time at a regular pace (or equivalently gets divided by two/three/four... at a regular pace).

Example of exponential growth:

Example of exponential decay: Everyone in the class takes a coin. We all toss the coin together. At every coin toss, the people who get "tail" stop. People who get "head" continue on. Completing the following table and graph we get:



Example of exponential decay in nature. Exponential decay often occurs in probabilistic systems where there is some chance of "decay" over a certain timescale. The most common example in Nature is radioactive decay. Many atoms are not "stable" atoms: they can, with a certain probability for a given time period, "decay" into another atom (i.e. change into another element). Some elements, however, are more unstable than others, and decay much more rapidly. This propensity to decay is often measured through a "half-life". By definition, the probability of decay is exactly 1/2 after one half-life. For example:

- The half-life of Carbon 14 (used for radioactive dating) is 5730 years. Carbon 14 decays into Nitrogen 14.
- The half-life of Plutonium 239 (nuclear waste product) is 24,110 years
- The half-life of Iodine 131 (other waste product) is 8 days

The amount of radioactive material left after a certain time is an exponential function of time:

Example 1: Suppose you collect pure Carbon 14 in a sealed jar now.

- What percentage of Carbon 14 is left after 5730 years?

- What percentage of Carbon 14 is left after 100 years?

- What percentage of Carbon 14 is left after a 100,000 years?

Example 2: Suppose you find a sealed jar and identify it contains x percent of Carbon 14 and Nitrogen 14. What do you need to do to find when it was sealed as a function of x?