AMS 3 Final

Name: ______________________________

Calculators are not allowed.

Read all the questions before you start working on any of them. Start with the ones you are most comfortable with, and continue with the other ones later. Always double-check your answers.

Note that this final is graded out of 110 points. Any points above 100 will count as extra credit.

Relax, and do your best!

PROBLEM 1: SHORT QUESTIONS. [30 POINTS] 1pt/question, partial credit when noted.

NP 1. What is the equation of the line perpendicular to $y = x$ which goes through the point $(-1, 2)$?

slope is -1

$y = -x + b$

$2 = -(-1) + b \Rightarrow b = 1 \rightarrow$

ANSWER: $y = -x + 1 = 1 - x$

Given the functions $f(x) = \ln(x)$ and $g(x) = e^{x-2}$

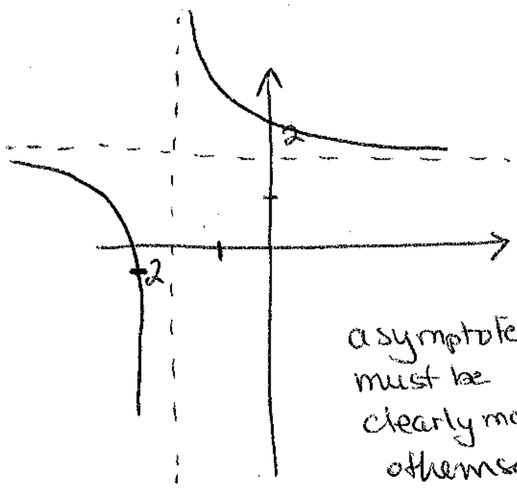
NP 2. What is the domain of f? $(0, +\infty)$

NP 3. What is $f \circ g(x)$? $\ln(e^{x-2}) = x - 2$ ← both ok

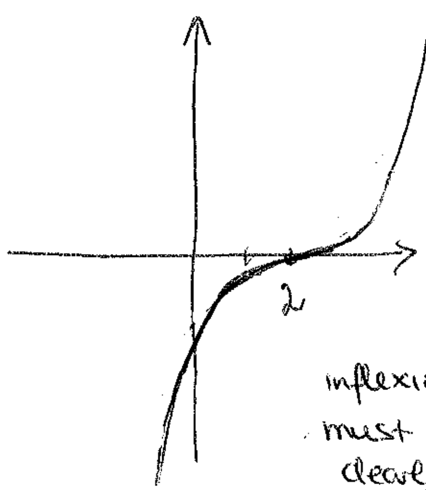
NP 4. What is $g \circ f(x)$? $e^{\ln x - 2} = \frac{e^{\ln x}}{e^2} = \frac{x}{e^2}$ ← both ok

NP 5. What is $\cos[\arccos(x+1)]$? $x+1$

6.7. Sketch the functions $f(x) = \frac{1}{x+2} + 2$ and $g(x) = (x-2)^3$



asymptotes must be clearly marked otherwise $-1/2$.



inflexion pt must be clearly marked otherwise $-1/2$

Given the parabola $y = x^2 + 6x - 1$: $(x+3)^2 - 9 - 1 = (x+3)^2 - 10$

NP 8. What are the coordinates of the vertex? $(-3, -10)$

NP 9. Does it open up or down? up

NP 10. What is the y-intercept? -1

11. What are the x-intercepts? $(x+3-\sqrt{10})(x+3+\sqrt{10})$

NP $-3+\sqrt{10}$, $-3-\sqrt{10}$ $\frac{-6 \pm \sqrt{40}}{2}$

12. Draw a signs table for the function $f(x) = \frac{3x^2}{(x-1)(x+3)}$

		-3		0		1	
$3x^2$	$+$		$+$	0	$+$		$+$
$x-1$	$-$		$-$		$-$	0	$+$
$x+3$	$-$	0	$+$		$+$		$+$
	$+$	∞	$-$	0	$-$	∞	$+$

← [1/2] if not all ∞ and 0 are there.

NP 13. Given the function $f(x)$ of question 12, what is the domain of $\ln(f(x))$?

$D = (-\infty, -3) \cup (1, +\infty)$

NP 14. Given the function $f(x)$ of question 12, what is the domain of $2^{f(x)}$?

$\mathbb{R} - \{-3, 1\}$

15. Expand the function $\ln[f(x)]$ of question 13 into a sum (or difference) of logarithms of the kind $\ln(ax+b)$.

$$\ln f(x) = \ln(3x^2) - \ln(x-1) - \ln(x+3)$$ ← [1/2] only

$$= 2\ln x + \ln 3 - \ln(x-1) - \ln(x+3)$$

NP 16. Simplify the expression $\frac{\ln(2^{x+1})}{\ln(2^{2x+1})}$

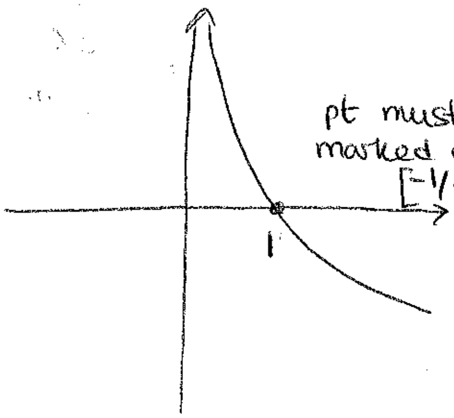
$\frac{x+1}{2x+1}$

NP 17. Simplify $f(x) = \frac{3^x 2^{-2x}}{9^{3x} 4^{-x}}$

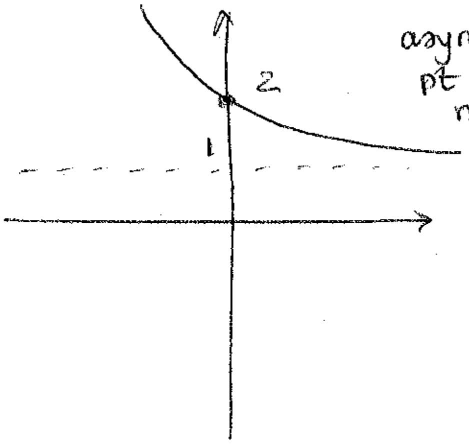
$$\frac{3^x \cancel{2^{-2x}}}{3^{6x} \cancel{2^{-2x}}} = 3^{-5x} = \frac{1}{3^{5x}}$$

NP 18. $\ln(x+y) = \ln(x)\ln(y)$: ~~TRUE~~ / FALSE

19. 20. Sketch the functions $-\log_2(x)$ and $e^{-x}+1$.



pt must be marked otherwise [-1/2].



asymptote and pt must be marked otherwise [-1/2].

21. Find the solution of the equation $e^{2x} - 2e^x - 3 = 0$

$y = e^x \qquad y^2 - 2y - 3 = 0 \qquad D = 4 + 12 = 16$

$y_{1,2} = \frac{2 \pm 4}{2} = 3, -1$

$e^x = 3 \Rightarrow x = \ln 3$

$e^x = -1 \Rightarrow$ no solution $\leftarrow$ [-1/2] if didn't note it's not a solution

$x = \ln 3$

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22. Solve the equation $\log_2(\log_3(x)) = -1$ (simplify your answer if needed)

$\log_3(x) = 2^{-1} = \frac{1}{2} \qquad x = 3^{1/2} = \sqrt{3}$

$x = \sqrt{3} = 3^{2^{-1}}$ [-1/2] if not simplified

NP 23. What is the inverse of the function $f(x) = 3x^{-1/\pi}$?

$y = 3x^{-1/\pi} \qquad \frac{y}{3} = x^{-1/\pi} \qquad \left(\frac{y}{3}\right)^{-\pi} = x$

$x = \left(\frac{3}{y}\right)^{\pi} \Rightarrow f^{-1}(x) = \left(\frac{3}{x}\right)^{\pi} = \left(\frac{x}{3}\right)^{-\pi} \leftarrow$ both ok.

NP 24. What is the inverse of the function $f(x) = 2^{2x}$

$y = 2^{2x}$

$\ln y = 2x \ln 2 \qquad x = \frac{\ln y}{2\ln 2} = \frac{\log_2(y)}{2}$

$f^{-1}(x) = \frac{\ln x}{2\ln 2} = \frac{\log_2(x)}{2}$ ← either ok $= \frac{\ln x}{\ln 4} = \log_4(x)$.

NP 25. Solve the equation $5^x = 3^{2x-1}$

$x \ln 5 = (2x-1)\ln 3$

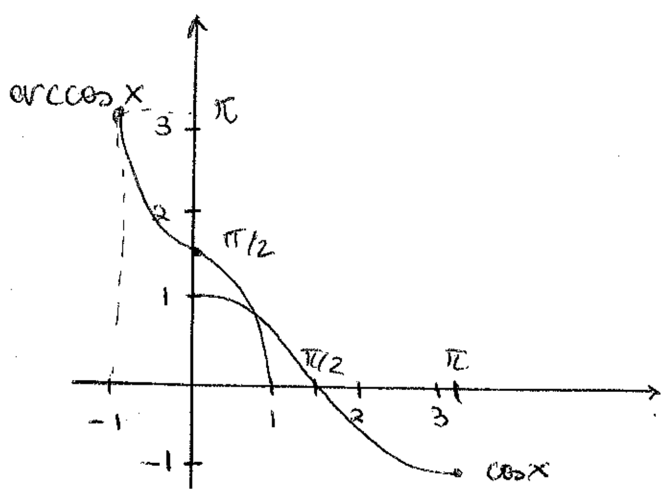
$x(\ln 5 - 2\ln 3) = -\ln 3 \qquad \ln 5 - 2\ln 3 = \ln\left(\frac{5}{9}\right)$

$X = \frac{-\ln 3}{\ln 5 - 2\ln 3} = \frac{-\ln 3}{\ln(5/9)}$ ← either ok

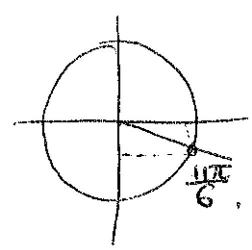
NP 26. Write $\left(\frac{1}{2}\right)^{t/\tau}$ as a natural exponential

$e^{t/2\, \ln(\frac{1}{2})} = e^{-\frac{t}{2}\ln 2}$

27.28. Sketch the function $\cos(x)$ and its inverse on the same graph, on the interval $[0, \pi]$



– ½ each case if π, $\frac{\pi}{2}$ not marked.



NP 29. What is $\sin\left(\frac{11\pi}{6}\right)$? $-1/2$

30. What is the amplitude, and period of the function $f(x) = 3\cos\left(\frac{2x}{3} + \frac{\pi}{6}\right)$.

amplitude = 3 [½]

period: $\frac{2\pi}{\frac{2}{3}} = 3\pi$ [½].

PROBLEM 2. INEQUALITIES. [15 POINTS]. Sketch the function

$$f(x) = \frac{2x}{x-2}$$

and solve the inequality $f(x) < 3$.

$\frac{2x}{x-2}$: as $x \to \infty$, $f(x) \to 2$

		0		2	
$2x$	$-$	0	$+$		$+$
$x-2$	$-$		$-$	0	$+$
	$+$	0	$-$	∞	$+$

Intersection:

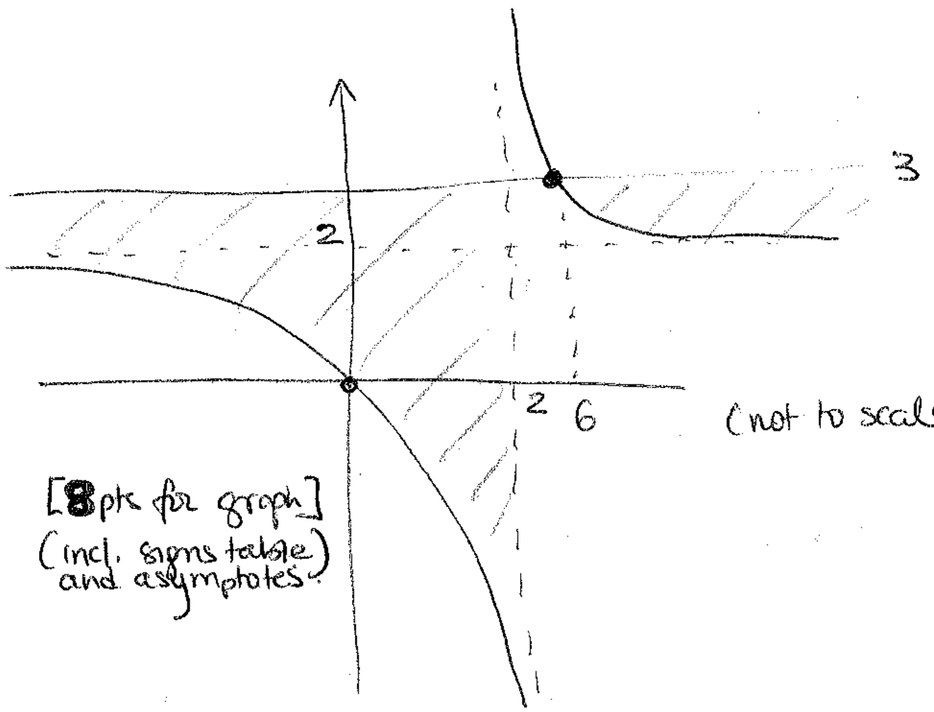
$$\frac{2x}{x-2} = 3$$

$$2x = 3(x-2)$$

$$2x = 3x - 6$$

$$x = 6$$

[3].



(not to scale

[8 pts for graph]
(incl. signs table and asymptotes)

ANSWER: ________________ [4].

$x \in (-\infty, 2) \cup (6, +\infty)$

PROBLEM 3: RATIONAL FUNCTIONS. [15 POINTS] Consider the function $f(x) = \frac{x^2-6x+8}{x^2-x}$.

(a) Factor this expression.

[3]

$$\frac{(x-2)(x-4)}{x(x-1)}$$

ANSWER: ______________________

[1] (b) What is the domain of $f(x)$? $D = \mathbb{R} - \{0, 1\}$

[2] (c) What are the x-intercepts? 2 and 4

[3] (d) Draw a signs table for $f(x)$

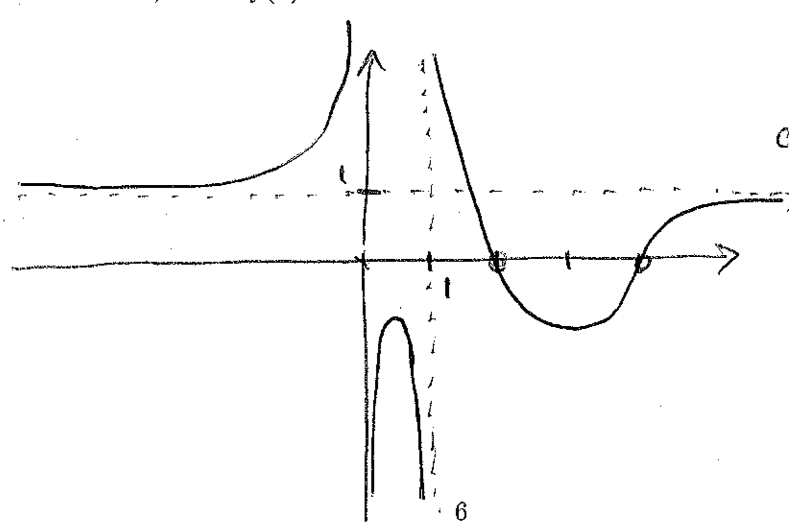
		0		1		2		4	
$x-2$	$-$		$-$		$-$	0	$+$		$+$
$x-4$	$-$		$-$		$-$		$-$	0	$+$
x	$-$	∞	$+$		$+$		$+$		$+$
$x-1$	$-$		$-$	∞	$+$		$+$		$+$
	$+$	∞	$-$	∞	$+$	0	$-$	0	$+$

(e) What is the behavior of $f(x)$ as x goes to $+\infty$ and $-\infty$?

[2] $f(x)$ has a horizontal asymptote at $y = 1$

(f) Using this information, sketch $f(x)$.

[4]



asymptotes must be clearly marked.

PROBLEM 4. LOGARITHMS AND EXPONENTIALS. [15 POINTS] We consider here the two functions

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

and

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

(a) Calculate cosh(0) and sinh(0).

$$\cosh(0) = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1$$

[2] cosh(0) = 1

$$\sinh(0) = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0$$

[2] sinh(0) = 0

(b) What are the domains of the functions $\cosh(x)$ and $\sinh(x)$?

(2) Domain of $\cosh(x)$: $\mathbb{R}$

[2] Domain of $\sinh(x)$: $\mathbb{R}$

(c) These functions have many properties which are reminiscent of the trigonometric functions $\cos(x)$ and $\sin(x)$. Here we prove one more of them. Evaluate and simplify the expression $\cosh^2(x) - \sinh^2(x)$, to show that it is actually constant. What is that constant?

[7]

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)$$

$$= \left(\frac{\cancel{e^x} + e^{-x} - \cancel{e^x} + e^{-x}}{2}\right)\left(\frac{e^x + \cancel{e^{-x}} + e^x - \cancel{e^{-x}}}{2}\right)$$

$$= \frac{\cancel{2}e^{-x}}{\cancel{2}} \cdot \frac{\cancel{2}e^x}{\cancel{2}} = e^{x-x} = 1$$

ANSWER: $\cosh^2(x) - \sinh^2(x) =$ 1 [2]

PROBLEM 5. TRIGONOMETRIC FUNCTIONS. [15 POINTS]

(a) Use the following addition formula

$$\sin(a-b) = \sin(a)\cos(b) - \sin(b)\cos(a)$$

simplify the expression

$$\frac{\sin(a-b)}{\cos(a)\cos(b)} + \frac{\sin(b-c)}{\cos(b)\cos(c)} + \frac{\sin(c-a)}{\cos(c)\cos(a)}$$

Note: show your full work with every step of the derivation, no points are awarded for guesses, or incomplete calculations.

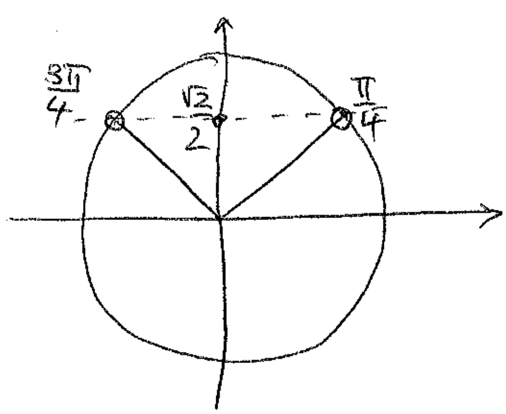
$$\frac{\sin a \cos b - \sin b \cos a}{\cos a \cos b} + \frac{\sin b \cos c - \sin c \cos b}{\cos b \cos c} + \frac{\sin c \cos a - \sin a \cos c}{\cos c \cos a}$$

$$= \frac{\sin a \cancel{\cos b}}{\cos a \cancel{\cos b}} - \frac{\sin b \cancel{\cos a}}{\cancel{\cos a} \cos b} + \frac{\sin b \cancel{\cos c}}{\cos b \cancel{\cos c}} - \frac{\sin c \cancel{\cos b}}{\cancel{\cos b} \cos c} + \frac{\sin c \cancel{\cos a}}{\cos c \cancel{\cos a}} - \frac{\sin a \cancel{\cos c}}{\cancel{\cos c} \cos a}$$

$$= \tan a - \tan b + \tan b - \tan c + \tan c - \tan a = 0.$$

[7].

[8] (b) Find *all* the solutions to the equation $\sin(x) = \frac{\sqrt{2}}{2}$, and draw them on the unit circle.



$(+2k\pi)$.

Breakdown:

- [2] for circle
- [3], [3] for solutions (−1 if $2k\pi$ missing)

ANSWER: $\frac{\pi}{4} + 2k\pi$ and $\frac{3\pi}{4} + 2k\pi$

Applied Problem. [20 points]

Settlers who moved to Australia in the 19th century brought rabbits with them for food and sports-hunting. The introduction of this pest rapidly led to one of the worse ecological disasters ever recorded, with hundreds of plant and animal species brought to exctinction as a result.

The introduction of rabbits has been traced to a single person, Thomas Austin, who released 24 rabbits in the wild near his house in Victoria (South Australia) in 1859. This problem studies the consequences of his foolish act.

We will begin counting the number of months since the introduction of the 24 rabbits with the variable t. At $t = 0$, in January 1859, there are therefore 24 rabbits in Australia (let's say, 12 males, 12 females).

(a) Assuming that the rabbit population doubles every months, how many rabbits will there be after 1 month (February), 2 months (March) and 3 months (April)?

[1] February 1859: ____48____

[1] March 1859: ____96____

[1] April 1859: ____192____

(b) Generalize your finding to express your estimate of the rabbit population after t months as the function

$$N(t) = N_0 a^t$$

What is the value of N_0? What is value of a?

[2][2] ANSWER: $N_0 =$ ____24____, $a =$ ____2____

(c) Using a change of base formula, express the function $N(t)$ as the following exponential in base e:

$$N(t) = N_0 e^{rt}$$

What is the value of r ? (Note: there is a table of values of ln below)

[3] ANSWER: $N(t) =$ ________, $r =$ ____$\ln 2$____

(d) Estimate how long it takes for the population to grow by a factor of 10. (Hint: find the time t for which $N(t) = 10N_0$ in terms of values of ln, and then estimate the result rounding to the nearest integer).

$$10 \cancel{N_0} = \cancel{N_0} (2^t)$$

$$10 = 2^t$$

$$\ln 10 = t \ln 2$$

$$t = \frac{\ln 10}{\ln 2} \simeq \frac{2.3}{0.7} \simeq 3 \text{ months}$$

[4] ANSWER: It takes 3 months months for the population to grow by a factor of 10.

(f) Deduce how long does it take for the population to grow by a factor of 10^9 (one billion)?

same thing with

$$10^9 \cancel{N_0} = \cancel{N_0}\, 2^t$$

$$9 \ln 10 = t \ln 2$$

$$t = 9\frac{\ln 10}{\ln 2} \simeq 9 \times 3 = 27$$

[4] ANSWER: It takes 27 months for the population to grow by a factor of a billion.

(g) Does your result support or contradict the following comment in Wikipedia: "In a classic example of unintended consequences, within ten years of the introduction in 1859, rabbits had become so prevalent that two million could be shot or trapped annually without having any noticeable effect on the population."

[2] ANSWER: yes

→ after 10 years, rabbit population exceeds many billions

→ killing 2 million is negligible on population

Useful numbers: $\ln(2) \simeq 0.7$, $\ln(3) \simeq 1.1$, $\ln(4) \simeq 1.4$, $\ln(5) \simeq 1.6$, $\ln(6) \simeq 1.8$, $\ln(7) \simeq 2$, $\ln(8) \simeq 2.1$, $\ln(9) \simeq 2.2$, $\ln(10) \simeq 2.3$.