ANSWERS

AMS 3: Midterm 2011

Name: ________________________________

Calculators are not allowed.

Read all the questions before you start working on any of them. Start with the ones you are most comfortable with, and continue with the other ones later. Always double-check your answers. If you need scrap paper, please ask the instructor/proctor.

Relax, and do your best!

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[40] PROBLEM 1: SHORT QUESTIONS. In the following questions, you are merely asked to provide the answer. No justification is needed. You should not be spending more than 1 minute per question.

[2] 1. What is the equation of the line passing through the points $(-1,2)$ and $(2,4)$? Write it in the form $y = ax + b$.

slope $= \frac{4-2}{2-(-1)} = \frac{2}{3}$ [1]

pt-slope formula: $y-2 = \frac{2}{3}(x+1)$ [1]

$y = \frac{2}{3}x + \frac{2}{3} + 2 = \boxed{\frac{2}{3}x + \frac{8}{3}}$

[2] 2. Find the linear function $f(x)$ such that $f(0) = 0$ and $f(1) = 4$.

$f(x) = 4x$ ([1] if they write $y=4x$)

Given the functions $f(x) = \sqrt{1-x}$ and $g(x) = \frac{1}{x}$

[1] if not simplified

[2] 3. Write down, and then simplify the expression $g(x)-g(x-1)$. $\frac{1}{x} - \frac{1}{x-1} = \frac{x-1-x}{x(x-1)} = \frac{-1}{x(x-1)}$

[2] 4. What is the domain of $f(x)$? $x \leq 1$ or $\mathcal{D} = (-\infty, 1]$ ← [1] if interval notation wrong

[2] 5. What is the domain of $g(x)$? $\mathbb{R} - \{0\}$ or $x \neq 0$ or $(-\infty, 0) \cup (0, +\infty)$

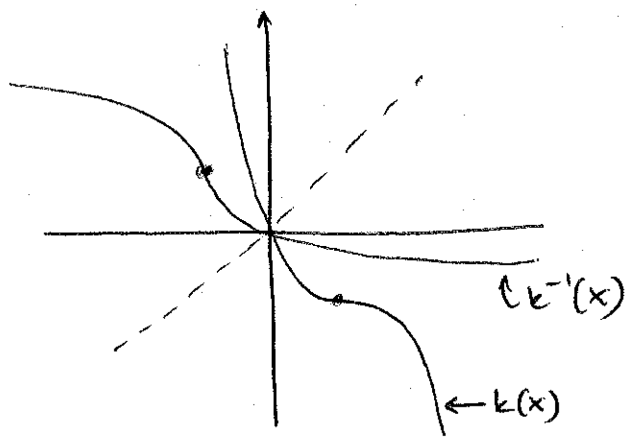
NP [2] 6. What is $f \circ g(x)$? $\sqrt{1-\frac{1}{x}}$

NP [2] 7. What is $g \circ f(x)$? $\frac{1}{\sqrt{1-x}}$

NP [2] 8. What is the inverse of $g(x)$? $g^{-1}(x) = 1/x$

+1 $y = \frac{1}{x}$ $x = \frac{1}{y}$ $f^{-1}(y) = 1/y$

[2][2] 9, 10. Sketch the function $k(x) = -(x-1)^3 - 1$ and its inverse on the same graph.



- shape must be roughly right.
- pts (1, -1) and (-1, 1) must be correct
- intersection must be on the --- line

NP [2] 11. If $f(x) = \frac{1}{x}$, what is $f[f^{-1}(2+3x)]$? $2+3x$

NP [2] 12. Complete the square for the expression $-x^2-2x+1$: ____

$$-(x^2+2x-1) = -(x^2+2x+1-1-1)$$
$$= -(x^2+2x+1-2)$$
$$= -[(x+1)^2-2] = -(x+1)^2+2 \quad \leftarrow \text{both forms ok}$$

Given the parabola $y = (x+4)^2 - 1$:

NP [2] 13. What are the coordinates of the vertex? $(-4, -1)$

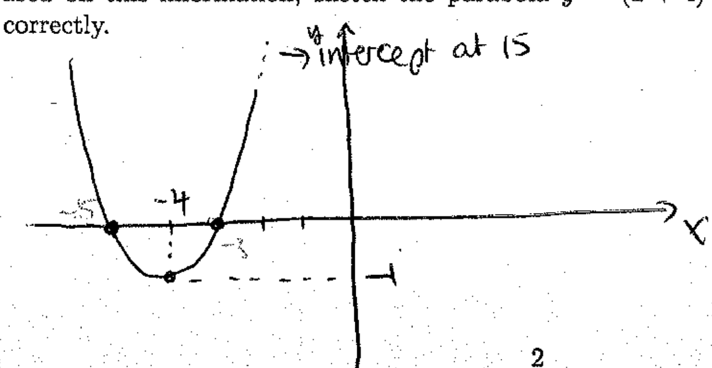
NP [2] 14. Does it open up or down? up

NP [2] 15. What is the y-intercept? 15

[2] 16. What are the x-intercepts? ____

$$(x+4)^2-1=0 \qquad (x+4)^2=1 \qquad x+4=\pm 1 \qquad x=\pm 1-4$$
$$= \begin{cases} -5 & [1] \\ -3 & [1] \end{cases}$$

[2] 17. Based on this information, sketch the parabola $y = (x+4)^2 - 1$, making sure to annotate your graph correctly.



NP [2] 18. How many times does this parabola intercept the x−axis?

$f(x) = x^2 + \sqrt{3}x - 3$? $D = 3 - 4(1)(-3) = 15 \rightarrow$ twice

[2] 19. Factor the quadratic $-x^2 + 2x + 5$

[1] for roots
[1] for factored form.

$$D = (2)^2 - 4(-1)(5) = 4 + 20 = 24$$

$$x_{1,2} = \frac{-2 \pm \sqrt{24}}{-2} = \frac{2 \pm \sqrt{24}}{2} = 1 \pm \sqrt{6}$$ ← both ok

$$-x^2 + 2x + 5 = -(x - 1 - \sqrt{6})(x - 1 + \sqrt{6}) = -\left(x - \frac{2+\sqrt{24}}{2}\right)\left(x - \frac{2-\sqrt{24}}{2}\right)$$

[2] 20. Factor the following expression by grouping, and make sure your result is *fully* factored: $x^4 - 2x^3 - 2x^2 + 4x$.

$$\begin{aligned} x^4 - 2x^3 - 2x^2 + 4x &= (x^4 - 2x^3) - (2x^2 - 4x) \\ &= x^3(x-2) - 2x(x-2) \\ &= (x^3 - 2x)(x-2) \\ &= \underbrace{x(x^2-2)(x-2)}_{[1]\text{ to get here}} = \underbrace{x(x-\sqrt{2})(x+\sqrt{2})(x-2)}_{[1]\text{ to finish}} \end{aligned}$$

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[20] PROBLEM 2: FUNCTIONS AND INVERSES: Consider the function $f(x) = x^2 - 2x + 1$.

NP [2] (a) What is the name of this kind of function? quadratic

NP [2] (b) What are the solutions to $f(x) = 0$?

$$x^2 - 2x + 1 = 0 \iff (x-1)^2 = 0 \iff \boxed{x = 1}$$

[4] (c) Factor $f(x)$, then draw a signs table for $f(x)$ $f(x) = (x-1)^2$

[2] for factored form
[2] for signs table

		1	
$x-1$	−	0	+
$x-1$	−	0	+
f	+	0	+

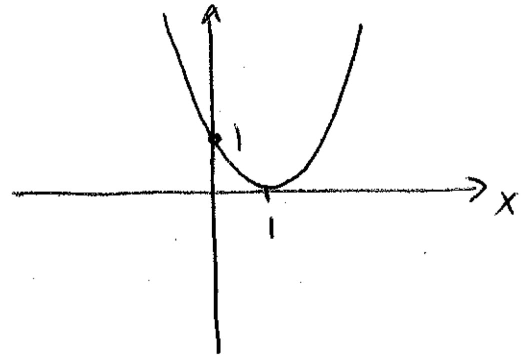
or

		1	
$(x-1)^2$	+	0	+
f	+	0	+

(both ok)

[2] (d) Using this information, sketch $f(x)$. Note: your y-intercept must be correct and annotated.

([-1] if y-intercept is missing)



y intercept is 1

[2] (e) Explain why, when finding the inverse, we should limit our study to the interval $x \geq 1$.

Needs to pass horizontal line test $\rightarrow$ need to chose $x \geq 1$

[2] f) Solve the equation $x^2 - 2x + 1 = y$ for x, and, by applying the confition $x \geq 1$, decide which of the two possible solutions to keep.

$$x^2 - 2x + 1 - y = 0$$

$$x_{1,2} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1-y)}}{2} = \frac{2 \pm \sqrt{4 - 4 + 4y}}{2}$$

$$= \frac{2 \pm \sqrt{4y}}{2} = 1 \pm \sqrt{y}$$

(answer from f to g must be consistent)

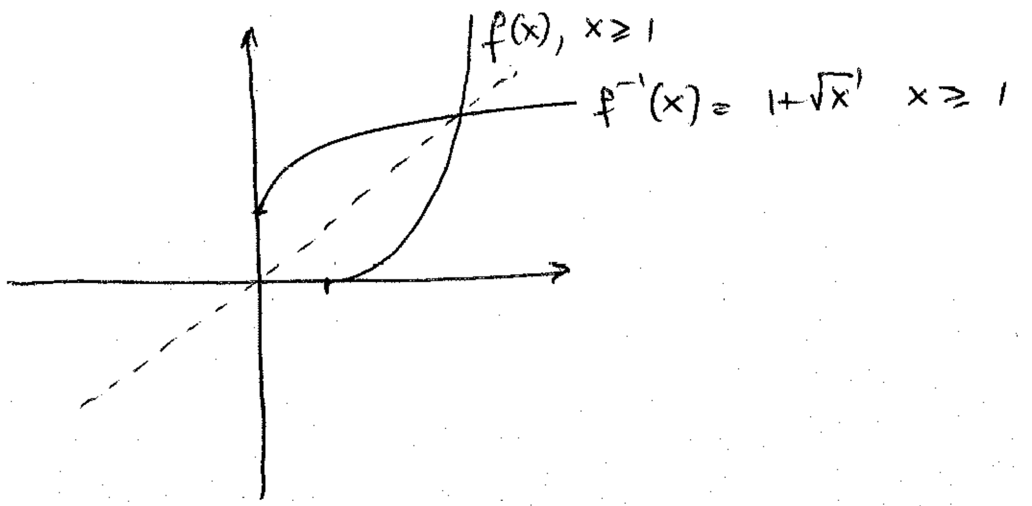
. if $x \geq 1$, need to keep $1+\sqrt{y}$ solution.

[1] (g) Deduce what $f^{-1}(x)$ is: $f^{-1}(x) = 1 + \sqrt{x}$

NP [1] (h) What is the domain of $f^{-1}(x)$? $x \geq 0$ or $[0, +\infty)$

[4] (i) Sketch the functions $f(x)$ (for $x \geq 1$) and $f^{-1}(x)$ (for x in the domain of f^{-1}) on the same graph. Clearly mark which is which.

([2] for each graph)



[20] PROBLEM 3. HIGHER ORDER POLYNOMIAL Consider the higher order polynomial function $x^3 - 3x^2 + 2x$.

(a) What is the behavior near $+\infty$ and $-\infty$?

[1] When x tends to $-\infty$, $f(x)$ goes to $-\infty$

[1] When x tends to $+\infty$, $f(x)$ goes to $+\infty$

[3] (b) Factor the function:

$$x(x^2 - 3x + 2) = x(x-2)(x-1)$$

(trial & error ok, or use quadratic formula).

[4] (c) Determine the $x-$ and $y-$ intercepts

$x-2=0 \Rightarrow \boxed{x=2}$

$x-1=0 \Rightarrow \boxed{x=1}$

$\boxed{x=0}$

Note: answer must be consistent with factored form.

[1] [1] [1] [1]

$x-$intercept(s): 0, 1, 2 $y-$intercept: 0

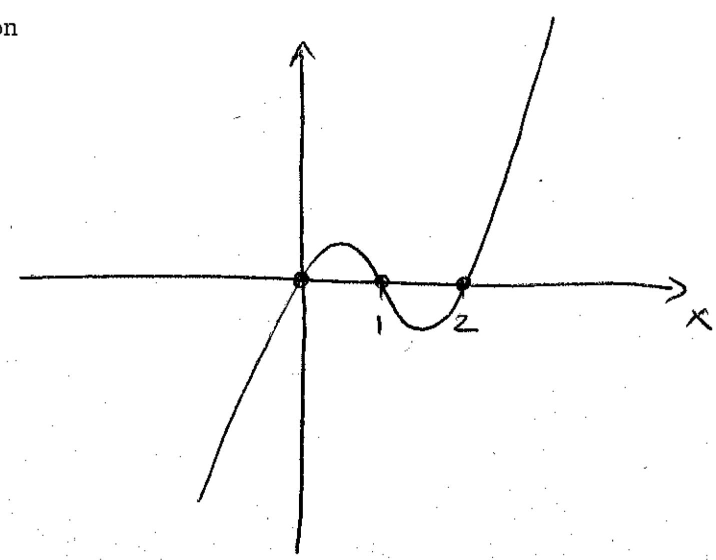
[5] (d) Draw a signs table

		0		1		2	
x	−	0	+		+		+
$x-1$	−		−	0	+		+
$x-2$	−		−		−	0	+
	−	0	+	0	−	0	+

[−1] if zeros are missing

[−1] per wrong line/column.

[6] (e) Sketch the function



[−1] per inconsistency with results above.

PROBLEM 4. APPLIED PROBLEM.

This problem guides you through a mathematical proof that the largest possible rectangular area of perimeter 1 (one) is achieved when that rectangular area is actually a square.

[4] **Question 1:** Considering a rectangle of length x and width y. How does y relate to x given the constraints of the problem?

$$\text{perimeter} = 1 = 2x + 2y$$
$$\Rightarrow \quad y = \frac{1-2x}{2} = \frac{1}{2} - x$$

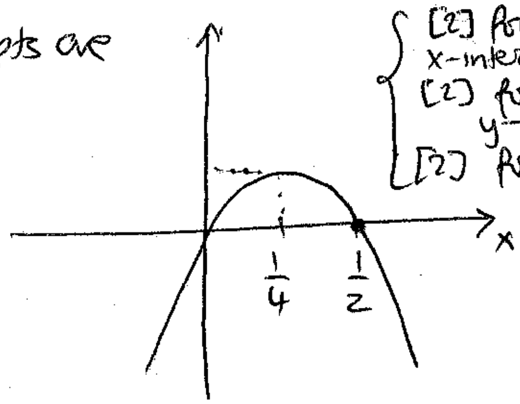
[2] **Question 2:** Show that the area of the rectangle as a function of x is $A(x) = \frac{1}{2}x - x^2$

$$\text{Area} = xy = x\left(\frac{1}{2} - x\right) = \frac{x}{2} - x^2$$

[2] **Question 3:** What is the name of this type of function? ____quadratic (polynomial ok)

[6] **Question 4:** Using the standard methods we have learned, sketch this function (Hint: find the x– and y–intercepts, etc..)

- $A(x) = x\left(\frac{1}{2} - x\right)$ so x-intercepts are
  $x = 0$
  $x = \frac{1}{2}$.
- parabola opens down
- y-intercept $= 0$



[2] for correct x-intercepts
[2] for correct y-intercept
[2] for open down

[4] **Question 5:** Using the standard methods we have learned, find the x–position of the maximum of this function.

- Maximum is at vertex. Vertex is at $x = -\frac{b}{2a} = \frac{-\frac{1}{2}}{2(-1)} = \boxed{\frac{1}{4}}$

OR $\frac{1}{2}x - x^2 = -\left(x^2 - \frac{1}{2}x\right) = -\left(\left(x - \frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right)$
$= -\left(x - \frac{1}{4}\right)^2 + \frac{1}{16} \Rightarrow \boxed{x = \frac{1}{4}}$ is max

either one fine, or just read from graph above

[2] **Question 6:** Calculate the y that corresponds to this optimal x. Why does this mean the rectangle is a square?

$$y = \frac{1}{2} - x = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$x = y$ so it's a square.