ANSWERS

AMS 3: Midterm 2011

Name: ______________________________

Calculators are not allowed.

Read all the questions before you start working on any of them. Start with the ones you are most comfortable with, and continue with the other ones later. Always double-check your answers. If you need scrap paper, please ask the instructor/proctor.

Relax, and do your best!

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[40] **Problem 1: Short questions.** In the following questions, you are merely asked to provide the answer. No justification is needed. You should not be spending more than 1 minute per question.

[2] 1. What is the equation of the line passing through the points $(1, 1)$ and $(-2, 2)$? Write it in the form $y = ax + b$.

Slope: $\frac{2-1}{-2-1} = \frac{1}{-3}$ [1]

pt slope formula: $y - 1 = -\frac{1}{3}(x-1)$

$y = 1 - \frac{1}{3}x + \frac{1}{3} = \boxed{-\frac{1}{3}x + \frac{4}{3}}$ [1]

[2] 2. Find the linear function $f(x)$ such that $f(0) = 0$ and $f(3) = 2$.

$f(x) = \frac{2}{3}x$ ([1] if they write $y = \frac{2}{3}x$)

Given the functions $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{1-x}$

[-1] if not simplified

[2] 3. Write down, and then simplify the expression $f(x) - f(x-1)$. $\frac{1}{x} - \frac{1}{x-1} = \frac{x-1-x}{x(x-1)} = \frac{-1}{x(x-1)}$

[2] 4. What is the domain of $f(x)$? $x \neq 0$ or $\mathcal{D} = (-\infty, 0) \cup (0, +\infty)$ or $\mathbb{R} - \{0\}$

[2] 5. What is the domain of $g(x)$? $x \leq 1$ or $(-\infty, 1]$ ← [-1] if interval notation wrong

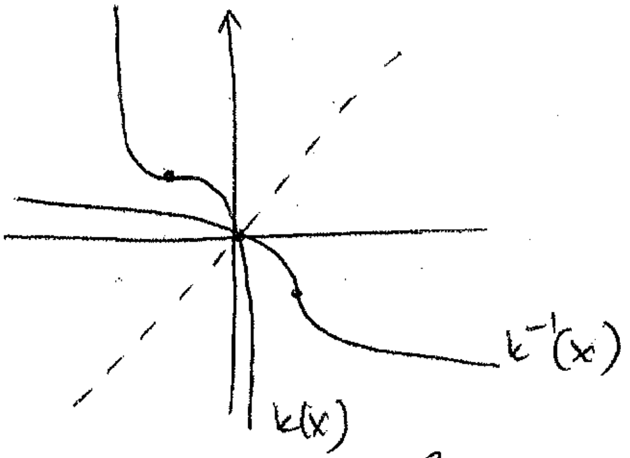
NP [2] 6. What is $f \circ g(x)$? $\frac{1}{\sqrt{1-x}}$

JP [2] 7. What is $g \circ f(x)$? $\sqrt{1 - \frac{1}{x}}$

X [2] 8. What is the inverse of $g(x)$? $g^{-1}(x) = 1 - x^2$

$y = \sqrt{1-x}$ $y^2 = 1 - x$ $x = 1 - y^2 = f^{-1}(y)$

[2],[2] 9, 10. Sketch the function $k(x) = -(x+1)^3 + 1$ and its inverse on the same graph.



· shape must be right
· positions of (-1, 1) and (1, -1) pts must be ok
· intersection of inverses must be on dashed line.

NP [2] 11. If $f(x) = \frac{1}{x}$, what is $f[f^{-1}(2x^2)]$? $2x^2$

NP [2] 12. Complete the square for the expression $-x^2+2x+3$:

$$-(x^2-2x-3) = -(x^2-2x+1-1-3)$$
$$= -((x-1)^2-4) = -(x-1)^2+4$$

Given the parabola $y = -(x+1)^2 + 4$:

NP [2] 13. What are the coordinates of the vertex? $(-1, 4)$

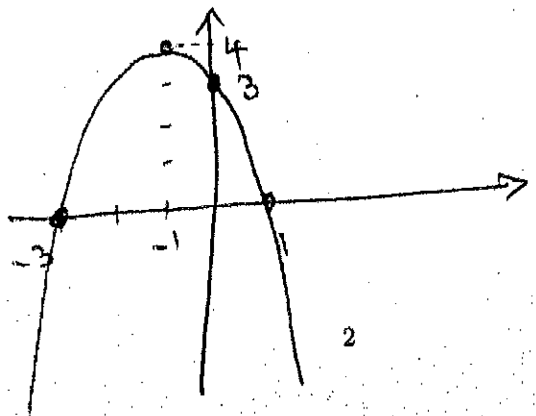
NP [2] 14. Does it open up or down? down

NP [2] 15. What is the y–intercept? 3

NP [2] 16. What are the x–intercepts?

$$-(x+1)^2 + 4 = 0 \Rightarrow (x+1)^2 = 4$$
$$\Rightarrow x+1 = \pm 2 \Rightarrow x = -1 \pm 2$$
$$\Rightarrow x = \begin{cases} -3 & [1] \\ 1 & [1] \end{cases}$$

[2] 17. Based on this information, sketch the parabola $y = -(x+1)^2 + 4$, making sure to annotate your graph correctly.



NP [2] 18. How many times does this parabola intercept the x—axis?

$f(x) = x^2 + \sqrt{3}x + 3$? $D = 3 - 4(1)(3) = 3 - 12 = -9 \to$ none.

[2] 19. Factor the quadratic $-x^2 + 2x + 5$

$D = (2)^2 - 4(-1)(5)$
$= 4 + 20 = 24$

[1] $x_{1,2} = \frac{-2 \pm \sqrt{24}}{-2} = \frac{2 \pm \sqrt{24}}{2} = 1 \pm \sqrt{6}$ ← both forms ok

[1] $-(x - 1 + \sqrt{6})(x - 1 - \sqrt{6}) = -\left(x - \frac{2+\sqrt{24}}{2}\right)\left(x - \frac{2-\sqrt{24}}{2}\right)$

[2] 20. Factor the following expression by grouping, and make sure your result is *fully* factored: $2x^4 - 4x^3 - x^2 + 2x$.

$(2x^4 - 4x^3) - (x^2 - 2x)$
$= 2x^3(x-2) - x(x-2) = (2x^3 - x)(x-2)$
$= x(2x^2 - 1)(x-2)$ [1]
$= x(\sqrt{2}x - 1)(\sqrt{2}x + 1)(x-2)$ [1].

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[20] PROBLEM 2: FUNCTIONS AND INVERSES: Consider the function $f(x) = x^2 - 2x + 1$.

NP [2] (a) What is the name of this kind of function? quadratic

NP [2] (b) What are the solutions to $f(x) = 0$?

$x^2 - 2x + 1 = 0$
$(x-1)^2 = 0 \to x = 1$

[4] (c) Factor $f(x)$, then draw a signs table for $f(x)$ $f(x) = (x-1)^2$

[2] for factored form
[2] for signs table

		1	
$x-1$	—	0	+
$x-1$	—	0	+
f	+	0	+

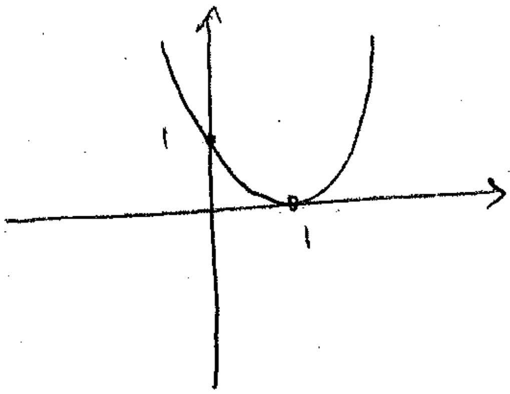
OR

		1	
$(x-1)^2$	+	0	+
f	+	0	+

both ok

[2] (d) Using this information, sketch $f(x)$. Note: your $y-$intercept must be correct and annotated.

[-1] if y intercept is missing



y intercept: $f(0) = 1$

[2] (e) Explain why, when finding the inverse, we should limit our study to the interval $x \geq 1$.

Needs to pass the horizontal line test → need to choose $x > 1$

[2] (f) Solve the equation $x^2 - 2x + 1 = y$ for x, and, by applying the confition $x \geq 1$, decide which of the two possible solutions to keep.

$$x^2 - 2x + 1 - y = 0$$

$$x_{1,2} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1-y)}}{2} = \frac{2 \pm \sqrt{4 - 4 + 4y}}{2}$$

$$= \frac{2 \pm \sqrt{4y}}{2} = 1 \pm \sqrt{y}$$

if $x \geq 1$, need $x = 1 + \sqrt{y}$

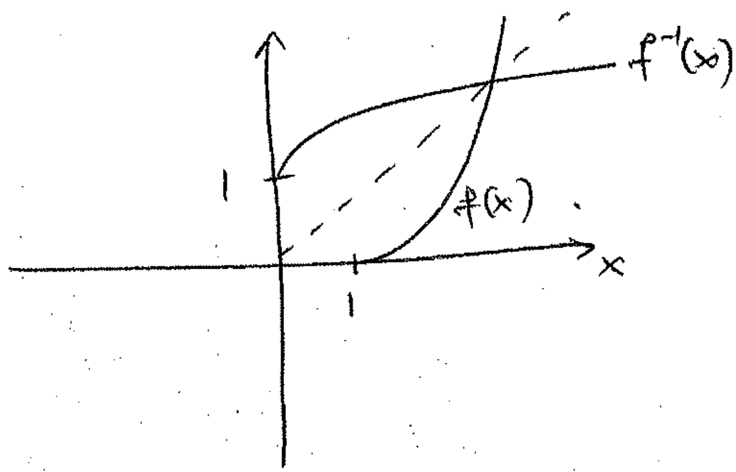
answers must be consistent

[1] (g) Deduce what $f^{-1}(x)$ is: $f^{-1}(x) = 1 + \sqrt{x}$

NP [1] (h) What is the domain of $f^{-1}(x)$? $x \geq 0$ or $[0, +\infty)$

[4] (i) Sketch the functions $f(x)$ (for $x \geq 1$) and $f^{-1}(x)$ (for x in the domain of f^{-1}) on the same graph. Clearly mark which is which.

([2] for each graph)



[20] PROBLEM 3. HIGHER ORDER POLYNOMIAL Consider the higher order polynomial function $-x^3+3x^2-2x$.

(a) What is the behavior near $+\infty$ and $-\infty$?

[1] When x tends to $-\infty$, $f(x)$ goes to $+\infty$

[1] When x tends to $+\infty$, $f(x)$ goes to $-\infty$

[3] (b) Factor the function:

$$-x(x^2-3x+2) = -x(x-1)(x-2)$$

trial error or use quadratic formula.

[4] (c) Determine the $x-$ and $y-$ intercepts

$x-2=0 \Rightarrow \boxed{x=2}$

$x-1=0 \Rightarrow \boxed{x=1}$

$\boxed{x=0}$

x–intercept(s): 0, 1, 2 [3] y–intercept: 0 [1]

[5] (d) Draw a signs table

		0		1		2	
$-x$	+	0	−		−		−
$x-1$	−		−	0	+		+
$x-2$	−		−		−	0	+
	+	0	−	0	+	0	−

[-1] if zeros are missing

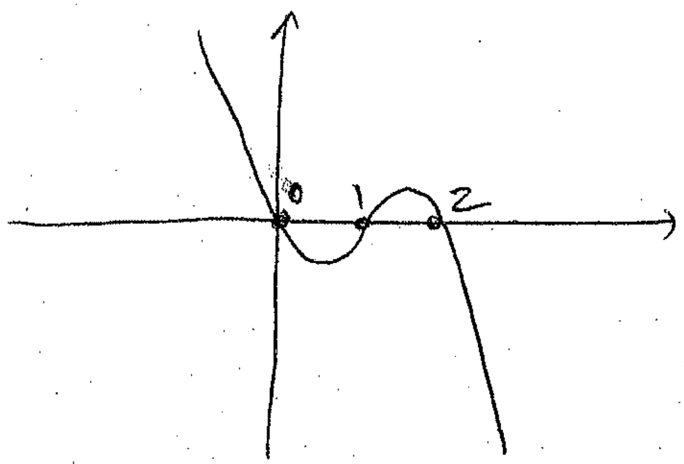
(-1) / wrong line/column.

Note:

		0		1		2	
-1	−		−		−		−
x	−	0	+		+		+
$x-1$	−		−	0	+		+
$x-2$	−		−		−	0	+
	+	0	−	0	+	0	−

also OK.

[6] (e) Sketch the function



[-1] / INCONSISTENCY W/ ABOVE

[20] PROBLEM 4. APPLIED PROBLEM.

This problem guides you through a mathematical proof that the largest possible rectangular area of perimeter 1 (one) is achieved when that rectangular area is actually a square.

[4] **Question 1:** Considering a rectangle of length x and width y. How does y relate to x given the constraints of the problem?

$$\text{perimeter} = 1 = 2x+2y$$
$$y = \frac{1-2x}{2} = \frac{1}{2} - x$$

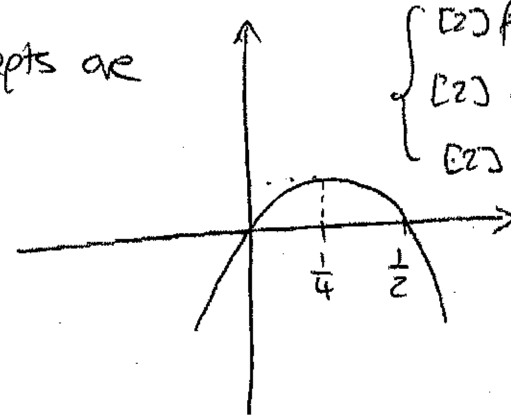
[2] **Question 2:** Show that the area of the rectangle as a function of x is $A(x) = \frac{1}{2}x - x^2$

$$\text{Area} = xy = x\left(\frac{1}{2} - x\right) = \frac{x}{2} - x^2$$

NP [2] **Question 3:** What is the name of this type of function? ___quadratic (polynomial ok)

[6] **Question 4:** Using the standard methods we have learned, sketch this function (Hint: find the x- and y-intercepts, etc..)

- $A(x) = x\left(\frac{1}{2} - x\right)$ so x-intercepts are
  $x = 0$
  $x = \frac{1}{2}$
- y-intercept is 0
- opens down



[2] for correct x-int
[2] for correct y-int
[2] for open down

[4] **Question 5:** Using the standard methods we have learned, find the x-position of the maximum of this function.

- Maximum is vertex, at $x = -\frac{b}{2a} = -\frac{\frac{1}{2}}{2(-1)} = \boxed{\frac{1}{4}}$

or $\frac{x}{2} - x^2 = -\left(x^2 - \frac{x}{2}\right) = -\left(\left(x-\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right) = -\left(x-\frac{1}{4}\right)^2 + \frac{1}{16}$

or Read on graph $\rightarrow$ vertex at $\boxed{\frac{1}{4}}$

all ok

[2] **Question 6:** Calculate the y that corresponds to this optimal x. Why does this mean the rectangle is a square?

$$y = \frac{1}{2} - x = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$x = y$ so it's a square.