Chapter 6

Trigonometric functions and periodic functions

In this final chapter of our course, we will learn about the three basic trigonometric functions, sine, cosine and tangent, as well as their use in geometry and in modeling oscillatory periodic functions.

6.1 The basic trigonometric functions

6.1.1 Case study: *How to measure the heights of trees?*

The research group of Prof. Gilbert at UC Santa Cruz studies (among other things) the ecology of trees. One of the most crucial part of this research is the acquisition of data on the growth rates of various species of trees, which involves measuring the heights of trees at regular intervals. Now, while it is easy to measure the height of young saplings, how does one measure the height of ancient redwoods? Climbing them and using a measuring tape is certainly not an option. As it turns out, this problem is actually very easy provided one knows a little bit about the basic trigonometric functions. Let's now study them, and revisit the problem shortly.

6.1.2 Mathematical corner: Sine, cosine and tangent in right-angle triangles

Sine, cosine and tangent functions are usually defined through their association with right-angle triangles:

The sine, cosine and tangent functions of various angles are easily computed using a calculator (also, see later for more). This knowledge can then be used to infer the size of one side knowing another side and an angle, as in the following examples.

EXAMPLE 1: What is the relationship between the height and the base of an equilateral triangle? Use this to deduce what the cosine of $60°$ is.

EXAMPLE 2: Consider the following right-angle triangle inscribed in the unit circle. Using the Pythagorean formula, show that, for any angle a, $\cos^2 a + \sin^2 a = 1$.

6.1.3 Case study: *How to measure the heights of trees? (part 2)*

We can now use what we have learned to measure the height of trees! Indeed, consider a tree, and walk a reasonable distance away from it so you can see the top. In as much as possible, try to do this horizontally (i.e. do not walk uphill or downhill). Measure the distance between where you are standing, and the base of the tree. Then, using a compass, measure the angle between the horizon and the top of the tree. We can then measure the tree height using:

For instance, what is the height of a redwood tree, if the angle measured 20 meters away from its base is 76°?

6.2 Degrees and radians

Textbook Section 5.1

6.2.1 Case study: *How to measure the radius of the Earth?*

The first person to measure the radius of the Earth was Eratosthenes, a Greek scholar who lived around 200BC. The method he used was very clever, but required a lot of patience. Today, we can use a very similar method but much more rapidly thanks to airplanes and the use of cellphones. To do so, fly to any city close to the equator, and ask one of your friends to fly to another city a known distance d away, also close to the equator. Quito (Ecuador) and Macapa (Brazil) are good examples, and are separated by a distance of about 3000km. When the Sun is directly above you, call them and ask what angle the sun makes with the vertical for them. Using a compass again, they can measure that angle. (For the case of Quito and Macapa, they would find that the angle is 26.5°). How can you use that information to measure the radius of the Earth?

6.2.2 Mathematical corner: Arc lengths, degrees and radians

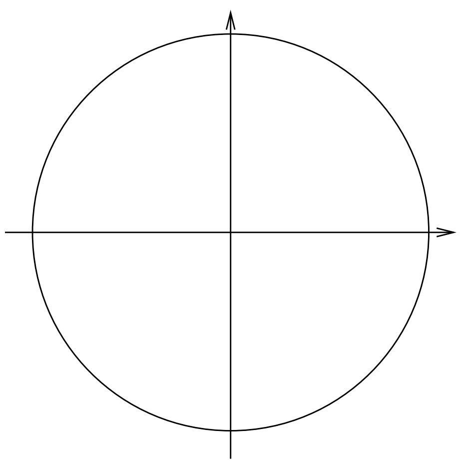
There are two major ways of measuring angles in geometry: in *degrees* and in *radians*.

The degree measure was introduced historically in astronomy to measure the displacements of stars, and is based on the fact that there are approximately 360 days in a year (well, there are in fact 365.25

days in a year, but 360 conveniently divides nicely by 2, 3, 4, 6, 10, 12, .., while 365.25 doesn't).

The radian measure is the one more commonly used in mathematics. It was introduced to solve problems very similar to the one shown in the case study above, and is based on the length of arcs of circles:

Based on this we have the following correspondance between degree and radians:



To summarize, to go between radians and degrees and vice-versa,

Note that while most calculators return the functions sine, cosine and tangent of an angle a, the user needs to input whether the angle is in degrees and radians.

Finally, note that for mathematical convenience angle are defined to be positive or negative depending on their direction:

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Also note that since the circle wraps around, an angle is always defined up to a value of 2π:

Now that we have introduced the concept of signed angles, we can return to the question of "What do the trigonometric functions look like?"

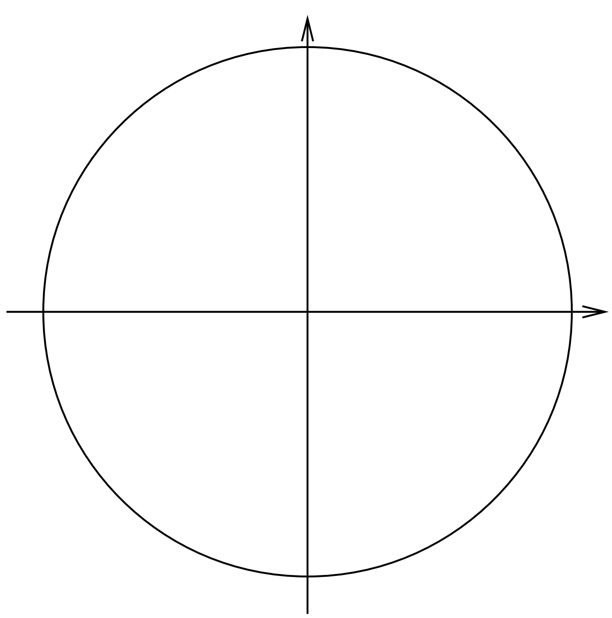
6.3 The unit circle, and the graphs of sine, cosine and tangent

Textbook Section 5.2

6.3.1 Construction of the unit circle

The unit circle is a wonderfully convenient way of *visualizing* the sine and cosine functions.

DEFINITION:



6.3.2 Sine and Cosine of important angles

Based on the graph of the unit circle, we can already deduce some particular values of the sine, cosine and tangent functions:

In addition to 0, $\pi/2$, π, $3\pi/2$ and 2π, there are 3 important angles for which you need to know the sine and cosine of:

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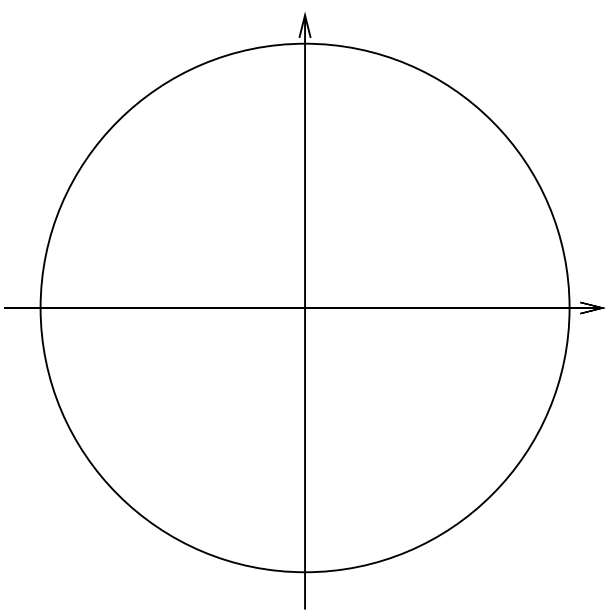
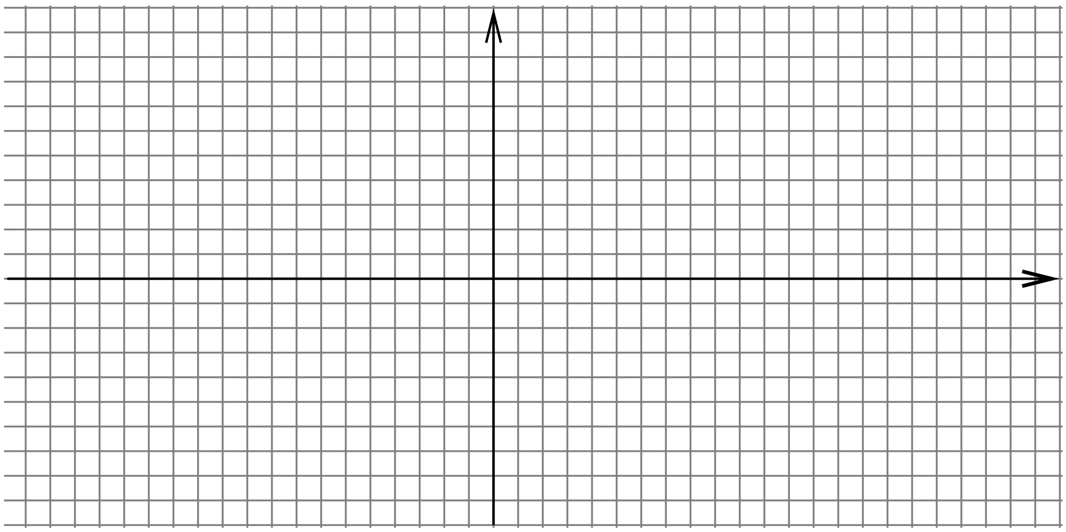
Based on the unit circle, we can now find the sine and cosine of many other angles:



Finally, we can use this information to plot the sine and cosine functions:



6.3.3 What can we deduce from the graphs of $\sin(x)$ and $\cos(x)$?

Based on the graphs of $\sin(x)$ and $\cos(x)$, we see that

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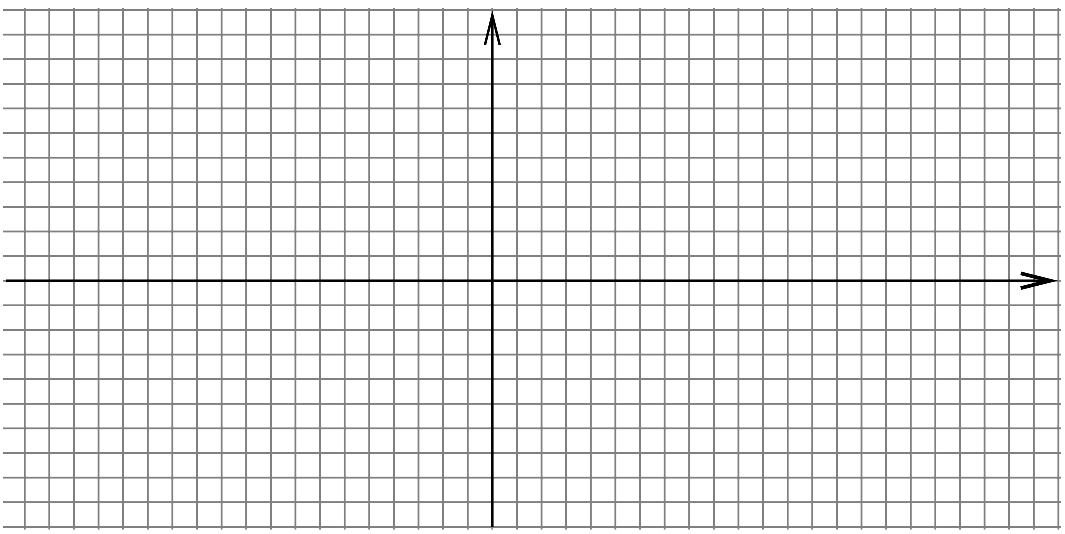
6.3.4 The graph of the tangent function

Textbook Section 5.5

We now look at the graph of the tangent function. By contrast with $\sin(x)$ and $\cos(x)$, $\tan(x)$ is not defined everywhere:

Let's use this information to graph $\tan(x)$.



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