Chapter 6

Trigonometric functions and periodic functions

In this final chapter of our course, we will learn about the three basic trigonometric functions, sine, cosine and tangent, as well as their use in geometry and in modeling oscillatory periodic functions.

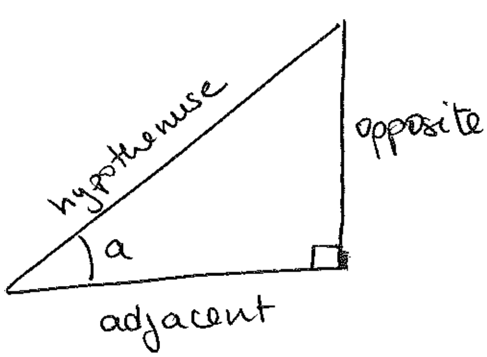
6.1 The basic trigonometric functions

6.1.1 Case study: *How to measure the heights of trees?*

The research group of Prof. Gilbert at UC Santa Cruz studies (among other things) the ecology of trees. One of the most crucial part of this research is the acquisition of data on the growth rates of various species of trees, which involves measuring the heights of trees at regular intervals. Now, while it is easy to measure the height of young saplings, how does one measure the height of ancient redwoods? Climbing them and using a measuring tape is certainly not an option. As it turns out, this problem is actually very easy provided one knows a little bit about the basic trigonometric functions. Let's now study them, and revisit the problem shortly.

6.1.2 Mathematical corner: Sine, cosine and tangent in right-angle triangles

Sine, cosine and tangent functions are usually defined through their association with right-angle triangles:

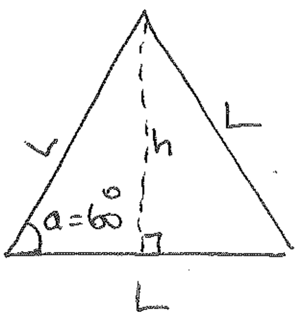


Given an angle a,

- the cosine of a is defined as
$$\cos a = \frac{\text{length of adjacent side}}{\text{length of hypothenuse}}$$
- the sine of a is defined as
$$\sin a = \frac{\text{length of opposite side}}{\text{length of hypothenuse}}$$
- finally: the tangent of a is
$$\tan a = \frac{\sin a}{\cos a} = \frac{\text{length of opposite side}}{\text{length of adjacent side}}.$$

The sine, cosine and tangent functions of various angles are easily computed using a calculator (also, see later for more). This knowledge can then be used to infer the size of one side knowing another side and an angle, as in the following examples.

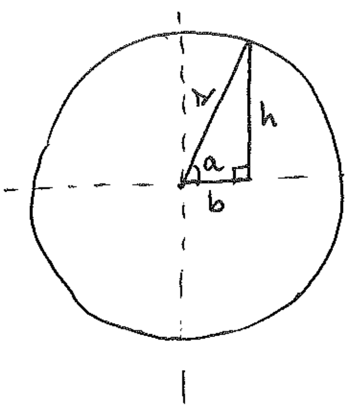
EXAMPLE 1: What is the relationship between the height and the base of an equilateral triangle?



- side length (of all 3 sides) is L → base length is also L
- Using the definition of the sine, we have

$\frac{h}{L} = \sin(60°)$ → use calculator to see that $h = \sin(60°) \cdot L = \frac{\sqrt{3}}{2} L \simeq 0.866 L$

EXAMPLE 2: Consider the following right-angle triangle inscribed on a circle of radius 1 ~~in the unit~~ circle. Using the Pythagorean formula, show that, for any angle a, $\cos^2 a + \sin^2 a = 1$.



height h is such that $\frac{h}{1} = \sin a$

base b is such that $\frac{b}{1} = \cos a$

so $h = \sin a$ and $b = \cos a$

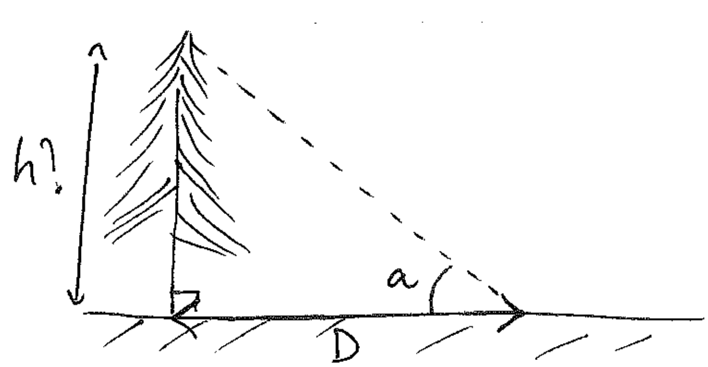
but using Pythagoras' theorem

$b^2 + h^2 = 1 \Rightarrow \boxed{\sin^2 a + \cos^2 a = 1}$

(for any a!)

6.1.3 Case study: *How to measure the heights of trees? (part 2)*

We can now use what we have learned to measure the height of trees! Indeed, consider a tree, and walk a reasonable distance away from it so you can see the top. In as much as possible, try to do this horizontally (i.e. do not walk uphill or downhill). Measure the distance between where you are standing, and the base of the tree. Then, using a compass, measure the angle between the horizon and the top of the tree. We can then measure the tree height using:



This is a right-angle triangle so using the definition of the tangent

$\frac{h}{D} = \tan a$ $\boxed{D, a \text{ are measured}}$

$\Rightarrow \boxed{h = D \tan a}$

For instance, what is the height of a redwood tree, if the angle measured 20 meters away from its base is 76°?

From the text, $D = 20$ and $a = 76°$ so

$$h = 20 \cdot \tan(76°) = 80.21 \text{ meters!}$$

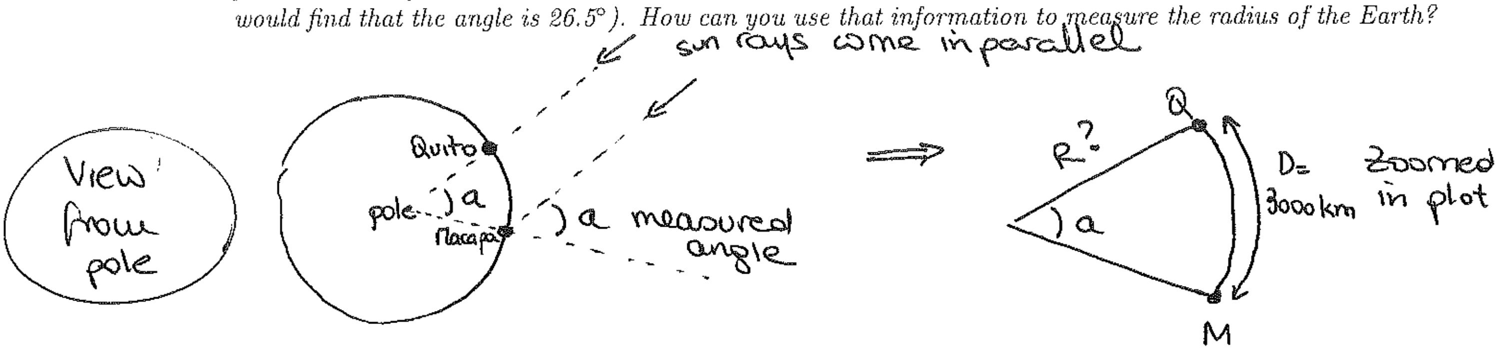
(Note: make sure your calculator knows a is in degrees)

6.2 Degrees and radians

Textbook Section 5.1

6.2.1 Case study: *How to measure the radius of the Earth?*

The first person to measure the radius of the Earth was Eratosthenes, a Greek scholar who lived around 200BC. The method he used was very clever, but required a lot of patience. Today, we can use a very similar method but much more rapidly thanks to airplanes and the use of cellphones. To do so, fly to any city close to the equator, and ask one of your friends to fly to another city a known distance d away, also close to the equator. Quito (Ecuador) and Macapa (Brazil) are good examples, and are separated by a distance of about 3000km. When the Sun is directly above you, call them and ask what angle the sun makes with the vertical for them. Using a compass again, they can measure that angle. (For the case of Quito and Macapa, they would find that the angle is 26.5°). How can you use that information to measure the radius of the Earth?



Can we use the measured distance D & angle a to get R?

→ Yes! We know that if the angle is 360° (all way around) then the distance is the Earth's circumference $2\pi R$

$\Rightarrow$ $360° \longleftrightarrow 2\pi R$

$a° = 26.5° \longleftrightarrow D = 3000\text{km}$

→ $360 \times 3000 = 26.5 \times 2\pi R$

so $R = \dfrac{360 \times 3000}{2\pi \times 26.5} = 6485$ km

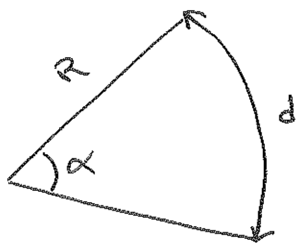
6.2.2 Mathematical corner: Arc lengths, degrees and radians

There are two major ways of measuring angles in geometry: in *degrees* and in *radians*.

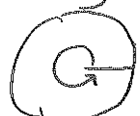
The degree measure was introduced historically in astronomy to measure the displacements of stars, and is based on the fact that there are approximately 360 days in a year (well, there are in fact 365.25

days in a year, but 360 conveniently divides nicely by 2, 3, 4, 6, 10, 12, .., while 365.25 doesn't).

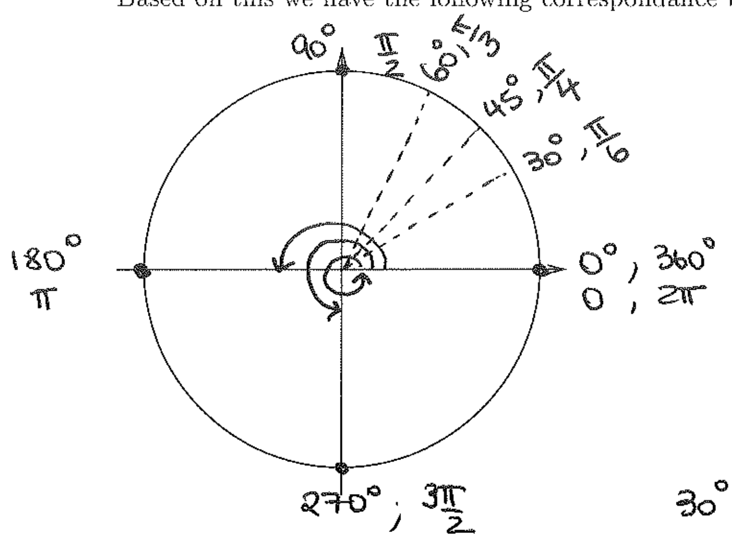
The radian measure is the one more commonly used in mathematics. It was introduced to solve problems very similar to the one shown in the case study above, and is based on the length of arcs of circles:



- if is in radians then $\boxed{d = R\alpha}$
- Going all around, we know that $d = 2\pi R$ → "360°" angle corresponds to 2π radians



Based on this we have the following correspondance between degree and radians:



$360° \longleftrightarrow 2\pi$ rad

$0° \longleftrightarrow 0$ rad

$180° \left(= \frac{360°}{2}\right) \longleftrightarrow \pi \quad \left(= \frac{2\pi}{2}\right)$

$90° \left(= \frac{180°}{2}\right) \longleftrightarrow \frac{\pi}{2} \quad \left(= \frac{\pi}{2}\right)$

$45° \left(= \frac{90°}{2}\right) \longleftrightarrow \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}$

$30° \longleftrightarrow \frac{\pi}{6} \qquad 60° \longleftrightarrow \frac{\pi}{3}$ etc...

To summarize, to go between radians and degrees and vice-versa,

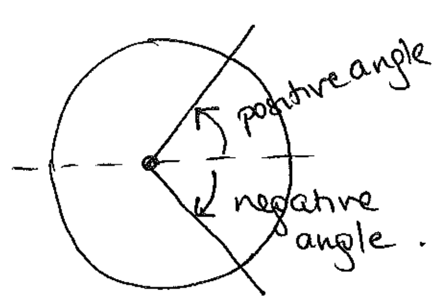
If a is in degrees, and α is the corresponding angle in radians, then

$$\boxed{\alpha = \frac{\pi}{180} a} \qquad \boxed{a = \frac{180}{\pi} \alpha}$$

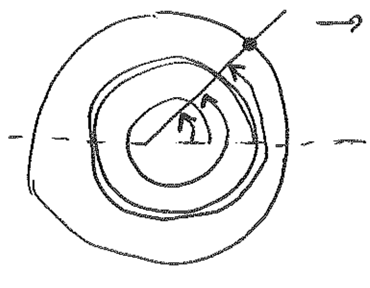
Note that while most calculators return the functions sine, cosine and tangent of an angle a, the user needs to input whether the angle is in degrees and radians.

Finally, note that for mathematical convenience angle are defined to be positive or negative depending on their direction:

- Positive angles go counterclockwise
- Negative angles go clockwise



Also note that since the circle wraps around, an angle is always defined up to a value of 2π:



→ angle α is the same as

- $\alpha + 2\pi$
- $\alpha + 4\pi$
- ⋮
- $\alpha - 2\pi$
- $\alpha - 4\pi$
- ⋮

} $\alpha + 2\pi n$ where n is any positive or negative number

Now that we have introduced the concept of signed angles, we can return to the question of "What do the trigonometric functions look like?"

6.3 The unit circle, and the graphs of sine, cosine and tangent

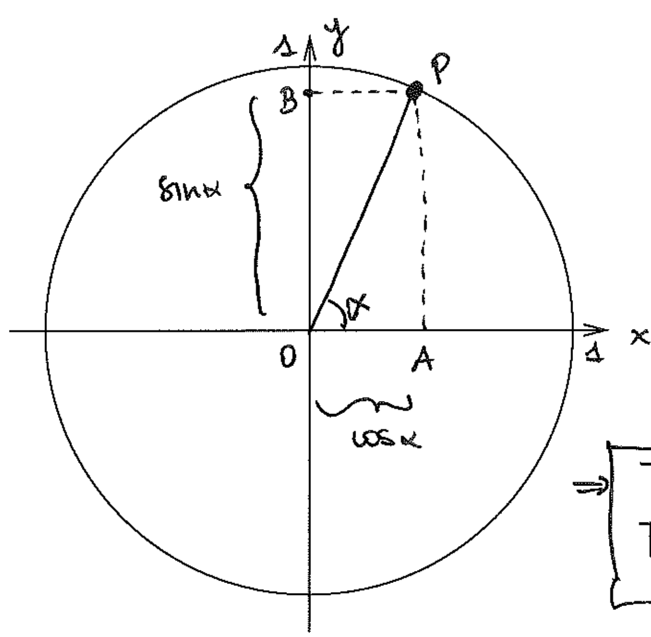
Textbook Section 5.2

6.3.1 Construction of the unit circle

The unit circle is a wonderfully convenient way of *visualizing* the sine and cosine functions.

Definition:

- The unit circle is the circle of radius 1, centred on the origin at (0,0)
- Any point P on the circle defines an angle x (in radians) (up to $2\pi n$)



- Length OP = 1 (by definition)
- $\frac{OA}{OP} = \cos\alpha \Rightarrow OA = \cos\alpha$
- $\frac{AP}{OP} = \sin\alpha \Rightarrow AP = \sin\alpha$

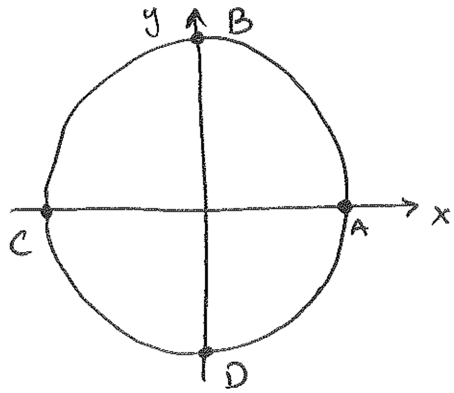
But OB = AP so OB = $\sin\alpha$

$\Rightarrow$ $\boxed{\text{The } x\text{-coordinate of P is } \cos\alpha \quad \text{The } y\text{-coordinate of P is } \sin\alpha}$

$\Rightarrow$ The unit circle can be used to visualize what $\cos\alpha$ & $\sin\alpha$ do as α increases (i.e as P goes around the circle).

6.3.2 Sine and Cosine of important angles

Based on the graph of the unit circle, we can already deduce some particular values of the sine, cosine and tangent functions:



A : $\alpha = 0$
→ $\cos(0)$ is x-coord of A
→ $\cos(0) = 1$
→ $\sin(0)$ is y-coord of A
→ $\sin(0) = 0$

B : $\alpha = \frac{\pi}{2}$ → $\cos\left(\frac{\pi}{2}\right) = 0$, $\sin\left(\frac{\pi}{2}\right) = 1$

C : $\alpha = \pi$ → $\cos(\pi) = -1$ $\sin(\pi) = 0$

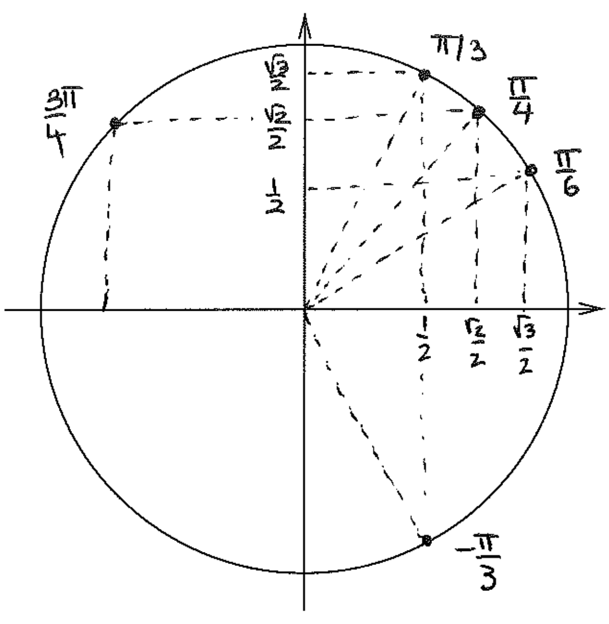
D : $\alpha = \frac{3\pi}{2}$ → $\cos\left(\frac{3\pi}{2}\right) = 0$, $\sin\left(\frac{3\pi}{2}\right) = -1$

In addition to 0, $\pi/2$, π, $3\pi/2$ and 2π, there are 3 important angles for which you need to know the sine and cosine of:

- $\frac{\pi}{6}$: $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$
- $\frac{\pi}{4}$: $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$
- $\frac{\pi}{3}$: $\cos\left(\frac{\pi}{3}\right) = 1/2$ $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

only 3 numbers to remember $\frac{1}{2}$, $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{3}}{2}$!

Based on the unit circle, we can now find the sine and cosine of many other angles:



- Start with upper right quadrant & fill up the known angles.
- Then use symmetries to discover the sine & cosine of many angles!

Examples:

- $\frac{3\pi}{4}$? → has same sine as $\frac{\pi}{4}$ → $\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$
  → has − cosine of $\frac{\pi}{4}$ → $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$
- $-\frac{\pi}{3}$? → has same cosine as $\frac{\pi}{3}$ → $\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$
  → has − sine of $\frac{\pi}{3}$ → $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

Finally, we can use this information to plot the sine and cosine functions:

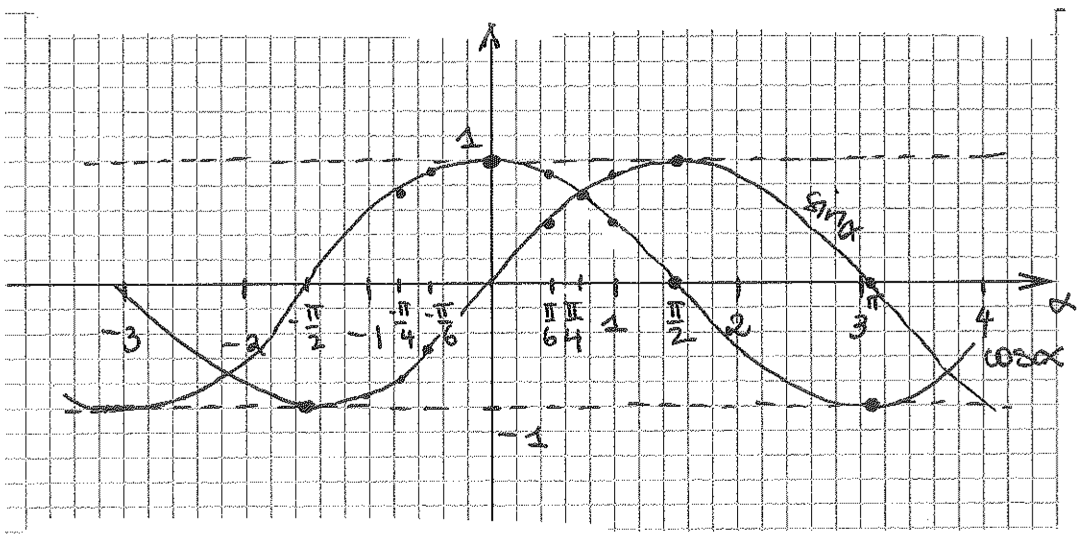
$\pi \simeq 3.1$

$\frac{\pi}{2} \simeq 1.5$

$\frac{\pi}{3} \simeq 1$

$\frac{\pi}{4} \simeq 0.75$

$\frac{\pi}{6} \simeq 0.5$



$\frac{\sqrt{2}}{2} \simeq 0.7$

$\frac{\sqrt{3}}{2} \simeq 0.85$

α:	$-\pi$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$\cos\alpha$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$
$\sin\alpha$	0	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$

→ Note that now, we can call α "x" as the argument of the functions.

6.3.3 What can we deduce from the graphs of $\sin(x)$ and $\cos(x)$?

Based on the graphs of $\sin(x)$ and $\cos(x)$, we see that

- Their domain is $\mathbb{R}$ and range is $[-1, 1]$
- The functions repeat themselves over an interval of 2π
- $\cos(x)$ is even and $\sin(x)$ is odd so $\cos(-x) = \cos(x)$ while $\sin(-x) = -\sin(x)$
- The graph of $\sin x$ is shifted to the right by $\frac{\pi}{2}$ from $\cos x$ so $\sin(x) = \cos(x - \frac{\pi}{2})$

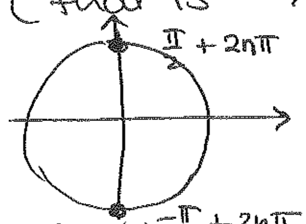
  (phase shift property)

6.3.4 The graph of the tangent function

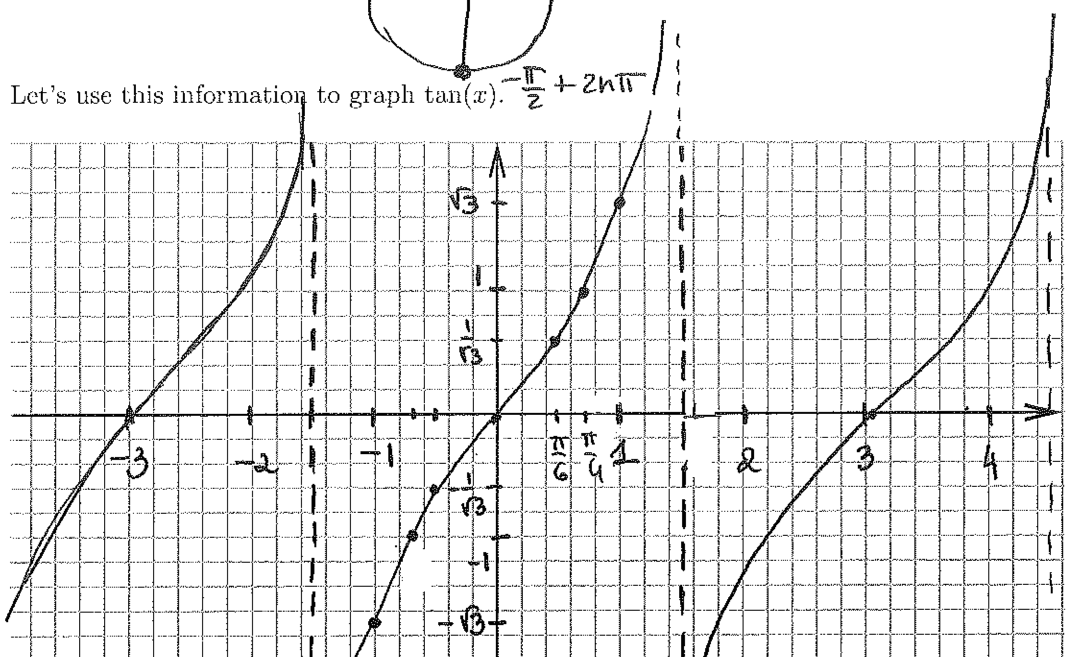
Textbook Section 5.5

We now look at the graph of the tangent function. By contrast with $\sin(x)$ and $\cos(x)$, $\tan(x)$ is not defined everywhere:

Indeed $\tan x = \dfrac{\sin x}{\cos x}$ → has an asymptote anytime $\cos x = 0$ (that is $x = \frac{\pi}{2}, \frac{3\pi}{2}, \dots -\frac{\pi}{2}, -\frac{3\pi}{2}, \dots$)



Let's use this information to graph $\tan(x)$.



$\sqrt{3} \simeq 1.7$

$\frac{1}{\sqrt{3}} \simeq 0.6$

x :	$-\pi$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
tanx :	0	$+\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

NOTES:

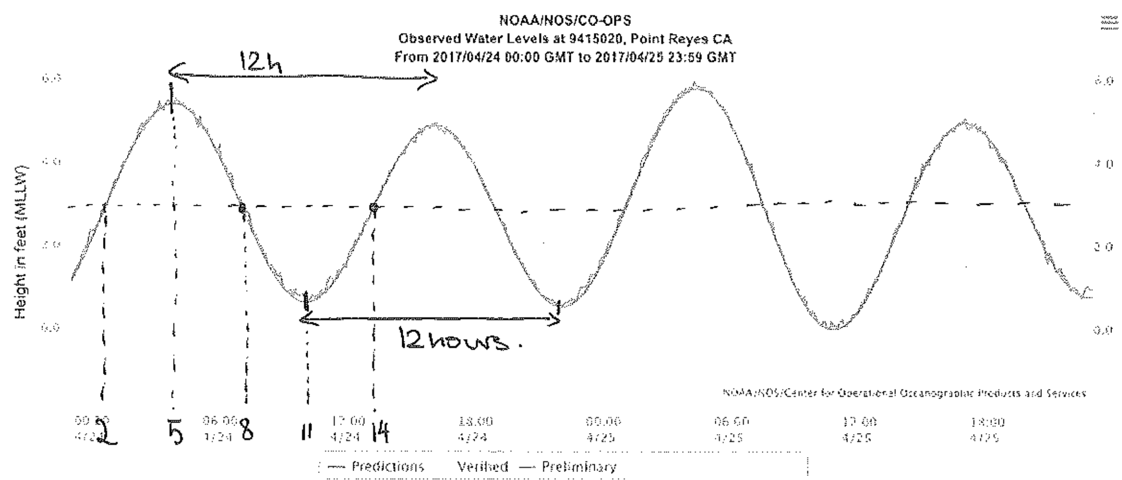
- The domain of $\tan x$ is $\mathbb{R} - \{\frac{\pi}{2} + 2n\pi, -\frac{\pi}{2} + 2n\pi\}$
- The range of $\tan x$ is $(-\infty, +\infty)$
- The $\tan x$ function is odd so $\tan(-x) = -\tan(x)$
- The $\tan x$ function has a basic pattern (between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ that repeats itself.

6.4 Periodic functions

Textbook Section 5.6

6.4.1 Case Study: *Tides*

The National Oceanic and Atmospheric Administration studies the tidally-induced variation of the water level with time in various coastal areas around the US, including Point Reyes in CA. They provide both forecasting services (i.e. predictions of the future water level) and monitoring services (i.e. measuring the actual water level). The following figure shows the result of one of their predictions and monitoring efforts, for the 48-hour period starting at midnight on April 24th, 2017.

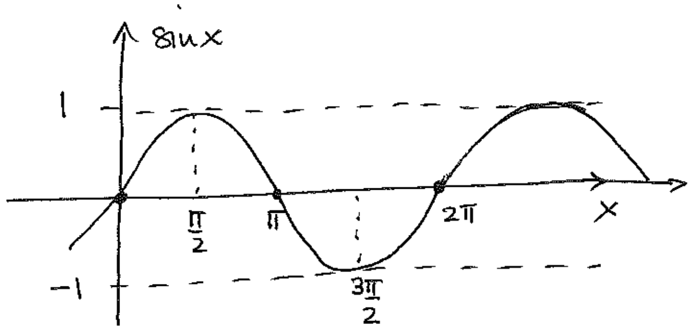


We see that

- Water level goes up & down in a sinusoidal pattern
- Peak & troughs every 12 hours or so, pattern repeats

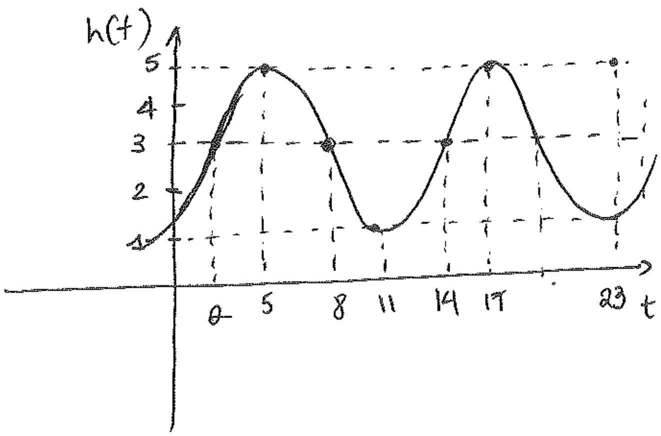
While it is actually quite difficult to model this phenomenon as accurately as NOAA did, we can try to model it approximately using sine or cosine functions. Let's relabel the hours starting at time $t = 0$, and increasing monotonically up to 48. Once that is done, how can we create a function that approximately models the data?

<u>Recall</u> Normal sine function



- oscillates between -1 and 1
- is equal to 0 at $0, \pi, 2\pi, \ldots$
- repeats every 2π
- oscillates around 0.
- has maxima at $\frac{\pi}{2}, \frac{5\pi}{2},$
- has minima at $\frac{3\pi}{2}, \ldots$

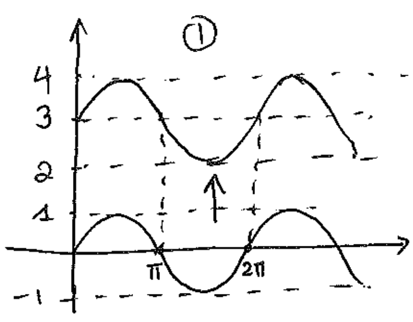
On the other hand here we have a function which is has the properties



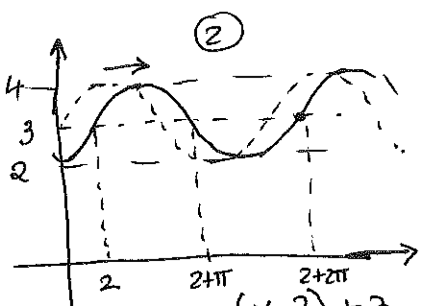
- oscillates around water height 3ft
- varies between ≃ 1 and ≃ 5 ft
- has maxima at 5, 17, ...
  minima at 11, 23, ...
- has zero at 2, 8, 14 ....

To model the tides function starting from the sine function, we therefore have to:

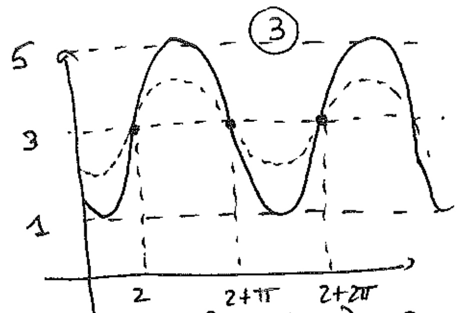
1. lift the sine function up by 3
2. move it to the right by 2
3. stretch it in the y direction by a factor of 2
- stretch it in the x-direction so the pattern has a length 12 instead of a length 2π.



$\sin x + 3$

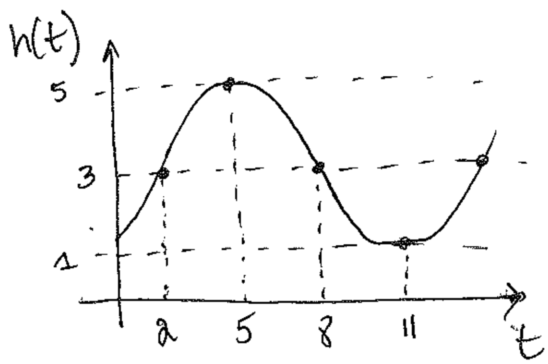


$\sin(x-2)+3$



$2\sin(x-2)+3$

4



$$h(x) = 2\sin\left[\frac{2\pi}{12}(x-2)\right]+3$$

or

$$h(t) = 2\sin\left[\frac{2\pi}{12}(t-2)\right]+3$$

$\Rightarrow$ as requested!

Using simple transformations on sine or cosine functions, we can therefore model many oscillatory functions. Let us now this how to do this in somewhat more generality.

6.4.2 Oscillatory functions

We now generalize what we saw in the previous section: the functions

$$f(x) = m + a \sin(b(x-c))$$
$$g(x) = m + a \cos(b(x-c))$$

are oscillatory functions where: a, b, c, m are real numbers

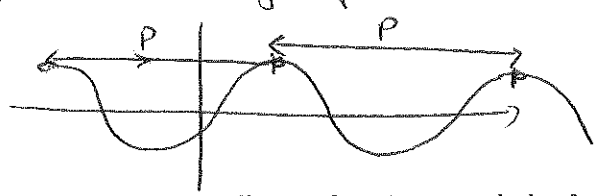
- m is the mean
- c is a shift
- a is the amplitude

These constants cause a vertical shift, a horizontal shift, and a vertical stretch of the function respectively. The interpretation of the constant b is a little less obvious, although we saw that it causes a horizontal stretching of the function. In fact, b is called the *frequency* of the oscillation, and it is related to the length of the pattern of oscillation, called the *period* of the oscillation.

DEFINITION: The period of an oscillation is the smallest non-zero number p such that $f(x+p) = f(x)$ for all x

Graphically, this says that the period of the function is the amount by which its graph has to be shifted in order to be itself:

shift by $\pm p$
recovers same graph!



To understand the relationship between the period of an oscillatory function p and the frequency b, note that

$$f(x+p) = f(x) \Rightarrow a \sin(b(x+p-c)) \cancel{+m} = a \sin(b(x-c)) \cancel{+m}$$
$$\Rightarrow \sin(bx+bp-bc) = \sin(bx-bc) \text{ for all } x.$$

Recall that $\sin(\alpha + 2\pi) = \sin \alpha$ for all α so

if $bp = 2\pi$, we indeed have $\sin(bx+2\pi-bc) = \sin(bx-bc)$

$$\Rightarrow bp = 2\pi \Rightarrow \boxed{p = \frac{2\pi}{b}}$$

The period of an oscillatory function is $\dfrac{2\pi}{\text{frequency}}$

EXAMPLE: What are the mean, amplitude, period, frequency and shift of the following functions:

- $f(x) = 2 + 2\cos(2x + 2)$ $= 2 + 2\cos(2(x+1))$

  mean : 2 frequency : 2 $\rightarrow$ period $= \frac{2\pi}{2} = \pi$
  amplitude : 2 shift : -1

- $f(x) = 3 + e\sin(\pi x)$

  mean 3 frequency π $\rightarrow$ period $= \frac{2\pi}{\pi} = 2$
  amplitude e shift : 0

- $f(t) = \sin(2\pi t - 1)$ $= \sin(2\pi(t - \frac{1}{2\pi}))$

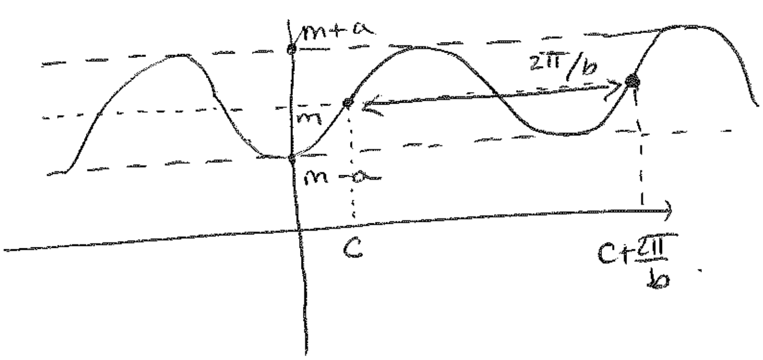
  mean 0 frequency : 2π $\rightarrow$ period : $\frac{2\pi}{2\pi} = 1$
  amplitude 1 shift : $+\frac{1}{2\pi}$

- $h(t) = 2\sin\left[\frac{2\pi}{12}(x - 2)\right] + 3$

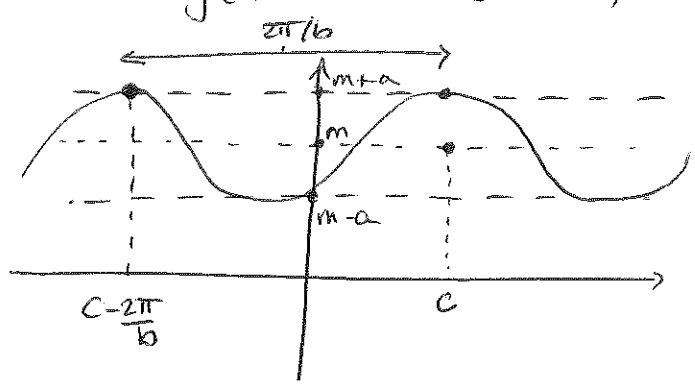
  mean : 3 frequency: $\frac{2\pi}{12}$ $\rightarrow$ period $= \frac{2\pi}{\frac{2\pi}{12}} = 12$
  amplitude : 2 shift 2

To summarize, here are the diagrams corresponding to the functions $f(x) = m + a\sin(b(x - c))$ and $g(x) = m + a\cos(b(x - c))$.

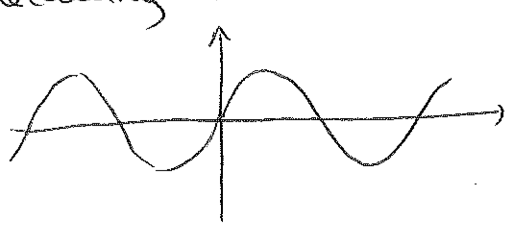
$f(x) = m + a\sin(b(x-c))$



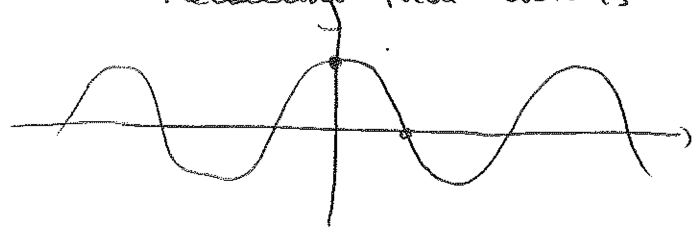
$g(x) = m + a\cos(b(x-c))$



Recalling that sin x is



Recalling that cos x is

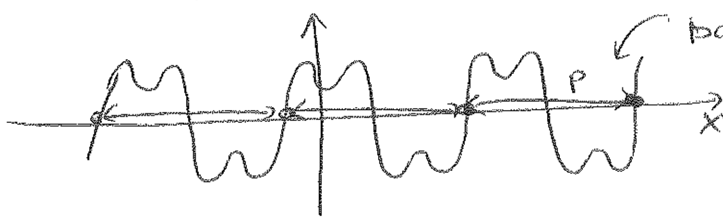


6.4.3 Periodic functions

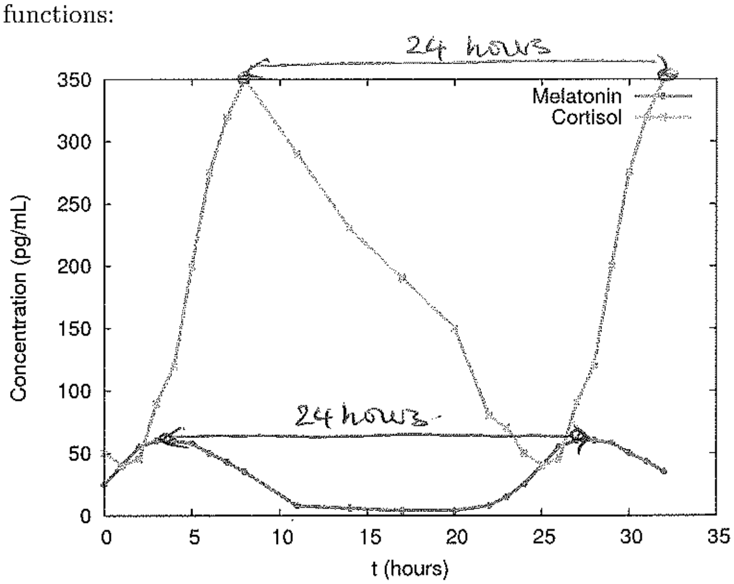
The definition of a period can be extended to any function that has a pattern that repeats itself over a certain period, even if that function does not arise from a sine or a cosine function.

DEFINITION: A periodic function is a function for which there exists a constant p $(\neq 0)$ such that

$$f(x+p) = f(x) \quad \text{for all } x$$



Here are some real-life examples of periodic functions that are more complex than simple sinusoidal functions:



Amount of melatonin/cortisol in blood controls circadian rythm, cycles of sleep & awake states
→ function is periodic with period 24 hours.

ECG voltage through heart as function of time, periodic with period of x seconds (= heartbeat time)

At rest $x \simeq 1$s.