5.3 The natural exponential and the natural logarithm

5.3.1 Case study: *Exponential population growth*

In ecology, it is quite common to attempt to model the evolution of the size of the population of a given species under various model assumptions. The simplest such model is called the Exponential growth model (for reasons that will become clear shortly) and the premise of this model is to assume that the population of the species studied multiplies by a given factor over a known reproduction time, without any deaths or loss of reproductivity. The multiplicative factor and the timescale are constant depend on the species, for instance

- *A pair of rabbits have on average 6 offsprings every 3 months.*
- *A pair of cats have on average 4 offsprings every 6 months.*
- *A pair of deer have on average 2 offsprings once a year.*

Based on this, what is the function that describes the evolution of a population of (1) rabbits with initially one breeding pair, (2) cats with initially 10 breeding pairs and (3) deers with initially 100 breeding pairs, as a function of the number x of months have have elapsed since?

We see that, indeed, each of these populations grows exponentially, hence the name of this population model. We also see in these three examples that different model assumptions on the initial number of individuals in the population, the reproductive timescales, and the reproductive habits (i.e. size of the litter) lead to different formulas. While we intuitively know that rabbits reproduce faster than cats, which themselves reproduce faster than deers, it is not necessarily easy to see this just based on the formulas we obtained. Indeed, we learned a while back that an exponential a^x *with a larger base* a *grows more rapidly, but how do we translate that knowledge when there is also a multiplicative factor in the exponent (e.g.* $x/4$, $x/6$, *etc..)? How do we compare the growth of two exponentials with different bases and different exponents?*

This is in fact a general problem with exponentials, namely, there is some degeneracy between the base, and any multiplicative factor in the exponent. Indeed, consider for instance:

In order to get rid of this modeling degeneracy and help compare different exponential models more easily, scientists have chosen one base in particular as the reference base for all exponentials. Curiously perhaps, it is not the exponential in base 2, but instead, it is the exponential in base e, and it is called the natural exponential. Let us now learn more about it, before revisiting the population growth problem.

5.3.2 Definitions

There is one particular base called the *natural* base for exponentials and logarithm.

DEFINITION:

The number e is a real number, with value approximately equal to:

The reason why this peculiar base is important in mathematics will be explored in more detail in Calculus.

Naturally, various function can be constructed from e^x:

DEFINITION:

PROPERTIES OF THE NATURAL LOGARITHM AND EXPONENTIAL: since these two functions are inverse of each other...

-
-
-

5.3.3 Changing from base a to the natural exponential

Textbook Section 4.5

As it turns out, in Mathematics we very rarely use anything other than the natural exponential and logarithm (with the exception, perhaps, of the logarithm in base 10). Instead, whenever we have a real-life problem that is modeled by an exponential that is not the natural exponential (see the various applications we did earlier), we transform it into the natural exponential using the Change of Base Rules. To change base from a base a exponential to the natural exponential (and vice versa, if needed):

The reason why this works is simple:

EXAMPLES:

- $2^x =$
- $\left(\frac{1}{4}\right)^x =$
- $2 \times 3^{-x} =$
- $4 \times 2^{-2x} =$

5.3.4 The growth and decay rates

Using the change of base formulas, we can ultimately express any growing or decaying exponential into one that is in base e:

In other words, our reference formula for exponentials can be rewritten as

where the plus or minus signs are used depending on whether the exponential is growing or decaying.

5.3.5 Case study: *Exponential population growth (part 2)*

Using this information, we can now re-cast each of the population growth models we have created into base e, which enables us to compare them better.

We now see that the respective growth rates of each population is

Not surprisingly, we recover our intuition that the growth rate of the rabbit population is larger than that of the cat population, which is in itself greater than the growth rate of the deer population.

We can also answer further questions such as how long does it take for each population to reach 1000 *individuals?*

Also, we can ask: how long does it take for the rabbit population to exceed the deer population?

Another common type of question may be "How long does it take for the cat population to double?" To answer this question, we have to solve the equation:

Interestingly, once we know the time T it takes to double the initial population, this is also the time it takes to double the population starting at any point of its evolution. Indeed, let's consider the size of the cat population at any time $x > 0$, and calculate the population size at time $x + T$:

This is in fact an important general property of exponentials.

5.3.6 The doubling and halving constants.

Consider a growing exponential $f(x) = be^{rx}$.

Note that if x is a time, the doubling constant is called the doubling time. If x is a length, then the doubling constant is called the doubling length. Similarly, if we consider a decaying exponential $f(x) = be^{-rx}$ then

The doubling (or halving) constant is intrinsically related to the growth (or decay) rate r. To see this, let us solve for the doubling constant:

We see that this constant is really constant (i.e it is independent of x).

Examples: What is the doubling time for the rabbit, cat and deer populations?

5.3.7 Changing from base a to the natural logarithm

As long as one always changes an exponential in base a to the natural exponential first, all the exponential equations can be solved using the natural logarithm, as we just have. This is why calculators rarely have anything but the natural logarithm (and sometimes the logarithm in base 10) on them. However, for completeness, note that there is also a rule to change a logarithm in base a to the natural logarithm, should this ever be needed:

The reason why this works is simple too:

EXAMPLES:

- $\log_2(x) =$
- $\log_{\frac{1}{4}} x =$

NOTE: This formula, when applied to a, yields the obvious relationship

If you are not sure of your change-of-base formula, this is a good way of double-checking that the formula you remember is the correct one.

This change of base is particularly useful because most calculators only provide $\ln(x)$ and not $\log_a(x)$. So, whenever you have to calculate $\log_a(x)$, you can use the formula to evaluate it using a normal calculator.

EXAMPLE:

- What is $\log_2(3)$?

- Solve the equation $2^x = 6$ and express the result as a natural logarithm.

- Show that for any a and b, the following is true: $\log_a(b) \log_b(a) = 1$.