5.3 The natural exponential and the natural logarithm

5.3.1 Case study: *Exponential population growth*

In ecology, it is quite common to attempt to model the evolution of the size of the population of a given species under various model assumptions. The simplest such model is called the Exponential growth model (for reasons that will become clear shortly) and the premise of this model is to assume that the population of the species studied multiplies by a given factor over a known reproduction time, without any deaths or loss of reproductivity. The multiplicative factor and the timescale are constant depend on the species, for instance

- *A pair of rabbits have on average 6 offsprings every 3 months.*
- *A pair of cats have on average 4 offsprings every 6 months.*
- *A pair of deer have on average 2 offsprings once a year.*

Based on this, what is the function that describes the evolution of a population of (1) rabbits with initially one breeding pair, (2) cats with initially 10 breeding pairs and (3) deers with initially 100 breeding pairs, as a function of the number x of months have have elapsed since?

(3): Deers: 100 deers = 50 pairs, each pair has 2 → 100 new ones,
→ population doubles every 12 mo
$N(x) = 200\,(2)^{\frac{x}{12}}$

(1) Rabbits

0 mo: 2 $\big) \times 4$
3 mo: 2+6 = 8
6 mo: 4 pairs; each pair has 6 → 24 more on top of existing 8 → total of 32
→ 32 = 8×4
⟶ basically the population multiplies by 4 every 3 months.

0 mo: 2 3 mo: 8 6 mo: 32 9 mo: 128 ...

→ $N(x) = 2 \cdot 4^{\frac{x}{3}}$ ← number of mo / reproductive cycle
← multiplier
↑ initial number
→ Check this is right!

(2) Cats

0 mo: 20 (10 pairs)
6 mo: Each pair has 4 offsprings → 40 more cats
→ total of 60 ⇒ multiply original by 3.
12 mo: 60 = 30 pairs ⟶ 30×4 offsprings = 120 new ones
→ add to 60 → 180 cats → multiply 60 by 3

→ each 6 mo we multiply by 3!

⟹ $N(x) = 20 \cdot 3^{\frac{x}{6}}$ → check.

We see that, indeed, each of these populations grows exponentially, hence the name of this population model. We also see in these three examples that different model assumptions on the initial number of individuals in the population, the reproductive timescales, and the reproductive habits (i.e. size of the litter) lead to different formulas. While we intuitively know that rabbits reproduce faster than cats, which themselves reproduce faster than deers, it is not necessarily easy to see this just based on the formulas we obtained. Indeed, we learned a while back that an exponential a^x with a larger base a grows more rapidly, but how do we translate that knowledge when there is also a multiplicative factor in the exponent (e.g. $x/4$, $x/6$, etc..)? How do we compare the growth of two exponentials with different bases and different exponents?

This is in fact a general problem with exponentials, namely, there is some degeneracy between the base, and any multiplicative factor in the exponent. Indeed, consider for instance:

We can write $4^x = (2^2)^x = 2^{2x}$

but also $4^x = (16^{\frac{1}{2}})^x = 16^{\frac{x}{2}}$

More generally, for any constant c,

$$a^x = (a^c)^{\frac{x}{c}} = (a^{\frac{1}{c}})^{cx}$$

The same function can be described by a different base/multiplicative factor pair.

In order to get rid of this modeling degeneracy and help compare different exponential models more easily, scientists have chosen one base in particular as the reference base for all exponentials. Curiously perhaps, it is not the exponential in base 2, but instead, it is the exponential in base e, and it is called the natural exponential. Let us now learn more about it, before revisiting the population growth problem.

5.3.2 Definitions

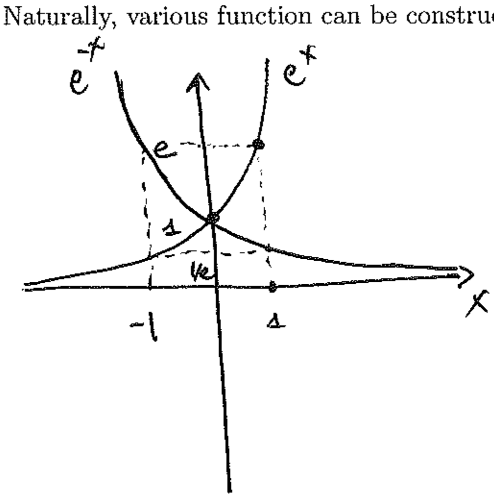
There is one particular base called the *natural* base for exponentials and logarithm.

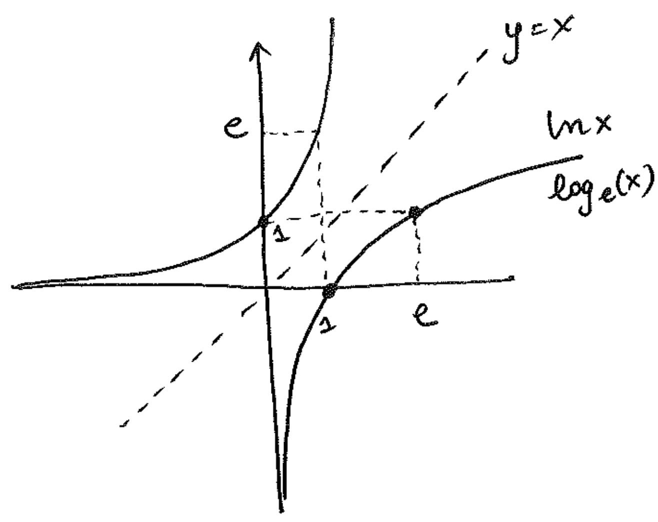
DEFINITION: The natural exponential is the exponential in base e: $f(x) = e^x$. ~~The natural log is the inverse of e^x; $\ln x$~~

The number e is a real number, with value approximately equal to: 2.71...

The reason why this peculiar base is important in mathematics will be explored in more detail in Calculus.

Naturally, various function can be constructed from e^x:





DEFINITION: The natural logarithm, $\ln(x)$, is the inverse of the natural exponential so if $y = e^x$ then $x = \ln y$

PROPERTIES OF THE NATURAL LOGARITHM AND EXPONENTIAL: since these two functions are inverse of each other...

- $e^{\ln x} = x$
- $\ln e^x = x$
- $\ln(1) = 0 \quad \ln(e) = 1$

Very important!!!

5.3.3 Changing from base a to the natural exponential

Textbook Section 4.5

As it turns out, in Mathematics we very rarely use anything other than the natural exponential and logarithm (with the exception, perhaps, of the logarithm in base 10). Instead, whenever we have a real-life problem that is modeled by an exponential that is not the natural exponential (see the various applications we did earlier), we transform it into the natural exponential using the Change of Base Rules. To change base from a base a exponential to the natural exponential (and vice versa, if needed):

Formula : $a^b = e^{b\ln a}$ (take base, & put its log in exponent)

→ works for any a & b!

The reason why this works is simple:

① $e^{b\ln a} = (e^{\ln a})^b = a^b$

② $a^b = e^{b\ln a} \Rightarrow \ln(a^b) = \ln(e^{b\ln a})$

$\Rightarrow b\ln a = b\ln a$ ← trivially true therefore original formula is correct.

EXAMPLES:

- $2^x = e^{x\ln 2}$
- $\left(\frac{1}{4}\right)^x = e^{x\ln(\frac{1}{4})} = e^{-x\ln 4}$
- $2 \times 3^{-x} = 2 \cdot 3^{-x} = 2 \cdot e^{-x\ln 3}$
- $4 \times 2^{-2x} = 4 \cdot 2^{-2x} = 4 \cdot e^{-2x\ln 2}$

5.3.4 The growth and decay rates

Using the change of base formulas, we can ultimately express any growing or decaying exponential into one that is in base e:

If $f(x) = b a^x$ then $f(x) = b e^{x \ln a}$

$f(x) = b a^{-x}$ then $f(x) = b e^{-x \ln a}$

In other words, our reference formula for exponentials can be rewritten as

$f(x) = b e^{rx}$ or $b e^{-rx}$ where r is any positive constant

where the plus or minus signs are used depending on whether the exponential is growing or decaying.

The constant r is called the growth rate if we have a growing exponential, and the decay rate if we have a decaying exponential

5.3.5 Case study: *Exponential population growth (part 2)*

Using this information, we can now re-cast each of the population growth models we have created into base e, which enables us to compare them better.

Rabbits : $N(x) = 2 \cdot 4^{\frac{x}{3}} = 2 \cdot e^{\frac{x}{3}\ln 4} \simeq 2e^{\cancel{1.38}x}$

Cats : $N(x) = 20 \cdot 3^{\frac{x}{6}} = 20\, e^{\frac{x}{6}\ln 3} \simeq 20\, e^{0.18x}$

Deer : $N(x) = 100\, 2^{\frac{x}{12}} = 100\, e^{\frac{x}{12}\cdot \ln 2} \simeq 100\, e^{0.057x}$

We now see that the respective growth rates of each population is

$r_{Rabbits} = \frac{1}{3}\ln 4 \simeq \cancel{1.38}$

bigger than

$r_{Cats} = \frac{\ln 3}{6} \simeq 0.18$

bigger than

$r_{deers} = \frac{\ln 2}{12} \simeq 0.057$

Not surprisingly, we recover our intuition that the growth rate of the rabbit population is larger than that of the cat population, which is in itself greater than the growth rate of the deer population.

We can also answer further questions such as how long does it take for each population to reach 1000 *individuals?* Solve $N(x) = 1000$!

Rabbits. $N(x) = 1000 \Rightarrow 2e^{\frac{x}{3}\ln 4} = 1000 \Rightarrow e^{\frac{x}{3}\ln 4} = 500$

$\rightarrow \ln\left(e^{\frac{x}{3}\ln 4}\right) = \ln(500) \rightarrow \frac{x}{3}\ln 4 = \ln 500$

$\rightarrow x = \frac{3\ln 500}{\ln 4} \simeq$ ~~424~~ 13.4 months

Cats: $20e^{\frac{x}{6}\ln 3} = 1000 \rightarrow e^{\frac{x}{6}\ln 3} = 50 \rightarrow \frac{x}{6}\ln 3 = \ln 50$

$\rightarrow x = 6\frac{\ln 50}{\ln 3} = 21.3$ months

Deers: $100e^{\frac{x}{12}\ln 2} = 1000 \rightarrow e^{\frac{x}{12}\ln 2} = 10 \rightarrow x = \frac{12\ln 10}{\ln 2} = 39.86$ mo

Also, we can ask: how long does it take for the rabbit population to exceed the deer population?

This time we want to know at which point the two populations are the same so we need to solve

$$2e^{\frac{x}{3}\ln 4} = 100e^{\frac{x}{12}\ln 2}$$

$$\rightarrow \frac{e^{\frac{x}{3}\ln 4}}{e^{\frac{x}{12}\ln 2}} = \frac{100}{2} = 50 \rightarrow e^{\frac{x}{3}\ln 4 - \frac{x}{12}\ln 2} = 50$$

$$\rightarrow \frac{x}{3}\ln 4 - \frac{x}{12}\ln 2 = \ln 50 \rightarrow x\left(\frac{\ln 4}{3} - \frac{\ln 2}{12}\right) = \ln 50$$

$$x = \frac{\ln 50}{\frac{\ln 4}{3} - \frac{\ln 2}{12}} = 9.6 \text{ mo}!$$

Another common type of question may be "How long does it take for the ~~rabbit~~ cat population to double?" To answer this question, we have to solve the equation:

Initially 20 ~~rabbits~~, so how long does it take until we have 40 cats? → solve

$$40 = 20e^{\frac{x}{6}\ln 3} \rightarrow 2 = e^{\frac{x}{6}\ln 3} \rightarrow \frac{x}{6}\ln 3 = \ln 2$$

$$\rightarrow x = 6\frac{\ln 2}{\ln 3} = 3.78 \text{ months}.$$

Interestingly, once we know the time T it takes to double the initial population, this is also the time it takes to double the population starting at any point of its evolution. Indeed, let's consider the size of the ~~rabbit~~ cat population at any time $x > 0$, and calculate the population size at time $x + T$:

$$N(x) = 20e^{\frac{x}{6}\ln 3} \rightarrow N(x+T) = 20e^{\frac{(x+T)}{6}\ln 3} \quad \text{where } T = \frac{6\ln 2}{\ln 3}$$

$$\rightarrow N(x+T) = 20e^{\frac{x}{6}\ln 3 + \frac{T}{6}\ln 3}$$

$$= 20e^{\frac{x}{6}\ln 3 + \frac{\cancel{6}\ln 2}{\cancel{6}\cancel{\ln 3}}\cancel{\ln 3}}$$

$$= 20e^{\frac{x}{6}\ln 3 + \ln 2} = 20 \cdot 2 \cdot e^{\frac{x}{6}\ln 3} = 2N(x)$$

This is in fact an important general property of exponentials.

5.3.6 The doubling and halving constants.

Consider a growing exponential $f(x) = be^{rx}$. The doubling constant is the value one must add to x in order to double $f(x)$:

$$f(x+c) = be^{r(x+c)} = 2f(x) = 2be^{rx}$$

Note that if x is a time, the doubling constant is called the doubling time. If x is a length, then the doubling constant is called the doubling length. Similarly, if we consider a decaying exponential $f(x) = be^{-rx}$ then The halving constant is the value one must add to x to halve $f(x)$:

$$f(x+c) = be^{-r(x+c)} = \frac{1}{2}f(x) = \frac{1}{2}be^{-rx}$$

The doubling (or halving) constant is intrinsically related to the growth (or decay) rate r. To see this, let us solve for the doubling constant:

$$\cancel{b}e^{r(x+c)} = 2\cancel{b}e^{rx} \Rightarrow e^{rx+rc} = 2e^{rx} \Rightarrow \cancel{e^{rx}} \cdot e^{rc} = 2\cancel{e^{rx}}$$

$$\rightarrow e^{rc} = 2 \rightarrow rc = \ln 2 \rightarrow \boxed{c = \frac{\ln 2}{r}}$$

We see that this constant is really constant (i.e it is independent of x). The halving constant is also $c = \frac{\ln 2}{r}$!

EXAMPLES: What is the doubling time for the rabbit, cat and deer populations?

Rabbits: $c = \frac{\ln 2}{r} = \frac{\ln 2}{\frac{\ln 4}{3}} = 3\frac{\ln 2}{\ln 4} = 1.5$ months

Cats: $c = \frac{\ln 2}{r} = \frac{\ln 2}{\frac{\ln 3}{6}} = 6\frac{\ln 2}{\ln 3}$ as we found earlier

Deers: $c = \frac{\ln 2}{r} = \frac{\ln 2}{\frac{\ln 2}{12}} = \frac{12\cancel{\ln 2}}{\cancel{\ln 2}} = 12$ as expected.

5.3.7 Changing from base a to the natural logarithm

As long as one always changes an exponential in base a to the natural exponential first, all the exponential equations can be solved using the natural logarithm, as we just have. This is why calculators rarely have anything but the natural logarithm (and sometimes the logarithm in base 10) on them. However, for completeness, note that there is also a rule to change a logarithm in base a to the natural logarithm, should this ever be needed:

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

The reason why this works is simple too: Take a power on both sides:

$$a^{\log_a(x)} = x \qquad a^{\frac{\ln x}{\ln a}} = e^{\frac{\ln x}{\cancel{\ln a}} \cdot \cancel{\ln a}} = e^{\ln x} = x$$

→ both sides are equal to x, and therefore equal to one another

EXAMPLES:

- $\log_2(x) = \frac{\ln x}{\ln 2}$
- $\log_{\frac{1}{4}} x =$ ________ $= \frac{\ln x}{\ln(1/4)}$

NOTE: This formula, when applied to a, yields the obvious relationship

$$\underbrace{\log_a(a)}_{1} = \underbrace{\frac{\ln a}{\ln a}}_{1} \leftarrow \text{obvious!}$$

If you are not sure of your change-of-base formula, this is a good way of double-checking that the formula you remember is the correct one.

This change of base is particularly useful because most calculators only provide $\ln(x)$ and not $\log_a(x)$. So, whenever you have to calculate $\log_a(x)$, you can use the formula to evaluate it using a normal calculator.

EXAMPLE:

- What is $\log_2(3)$?

$$\log_2(3) = \frac{\ln 3}{\ln 2}$$

- Solve the equation $2^x = 6$ and express the result as a natural logarithm.

Way #1: $\ln(2^x) = 6 \Rightarrow x\ln 2 = \ln 6 \Rightarrow x = \frac{\ln 6}{\ln 2}$

Way #2: $x = \log_2(6) \rightarrow x = \frac{\ln 6}{\ln 2}$

- Show that for any a and b, the following is true: $\log_a(b)\log_b(a) = 1$.

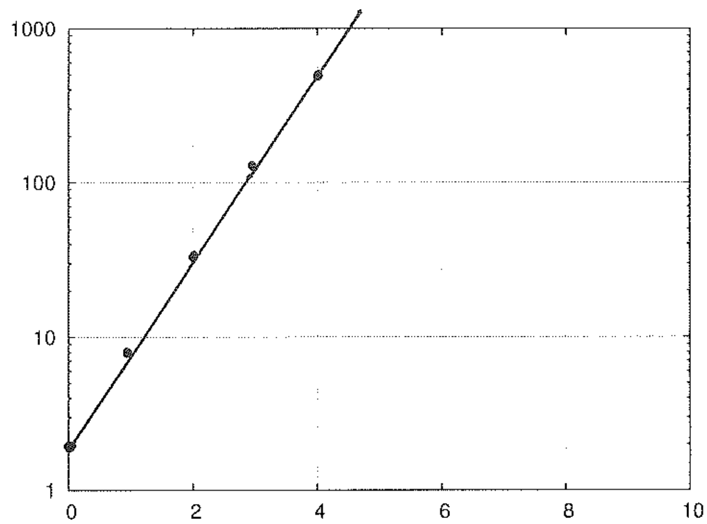
$$\log_a(b)\log_b(a) = \frac{\cancel{\ln b}}{\cancel{\ln a}} \cdot \frac{\cancel{\ln a}}{\cancel{\ln b}} = 1. \checkmark$$

5.4 Logarithmic axes

In the various case studies we have seen in the last 2 weeks, we came across logarithmic axes and their peculiar properties when it comes to graphing power laws and exponential functions. Let's see this once again through two examples.

Example 1: Graphing an exponential function on log-linear axes.

Let's draw the function $N(x) = 2 \times 4^x$ (the rabbit population growth model) on a graph with a logarithmic y-axis, and a linear x−axis (called a Log-Linear graph).

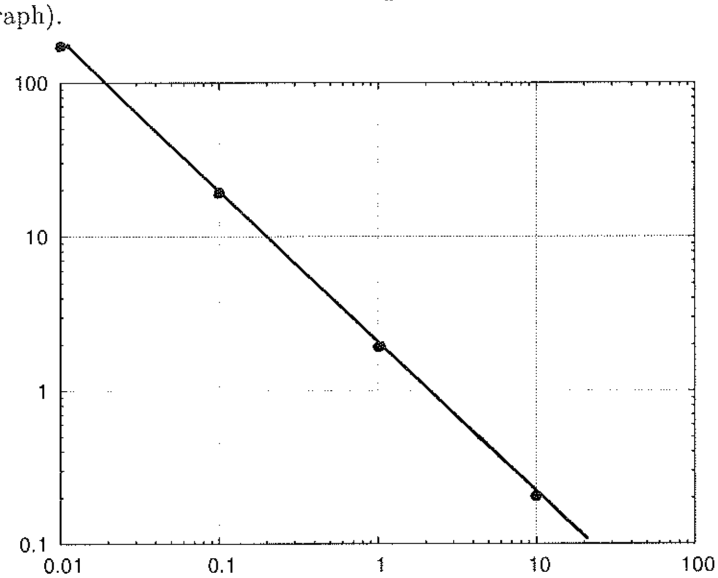


$N(0) = 2 \cdot 4^0 = 2$
$N(1) = 2.4^1 = 8$
$N(2) = 2.4^2 = 32$
$N(3) = 2.4^3 = 128$
$N(4) = 2.4^4 = 512$

→ all points lie on a straight line.

Example 2: Graphing a power law on log-log axes.

Let's draw the function $N(x) = \frac{2}{x}$ on a graph with a logarithmic x− and y-axes, (called a Log-Log graph).



$N(0.01) = \frac{2}{0.01} = 200$

$N(0.1) = \frac{2}{0.1} = 20$

$N(1) = \frac{2}{1} = 2$

$N(10) = \frac{2}{10} = 0.2$

$N(100) = \frac{2}{100} = 0.02$.

→ All points lie on a straight line.

These findings confirm what we found in earlier lectures, namely that

- An exponential function appears as a straight line on log-linear axes
- A power function " " " " on log-log axes.

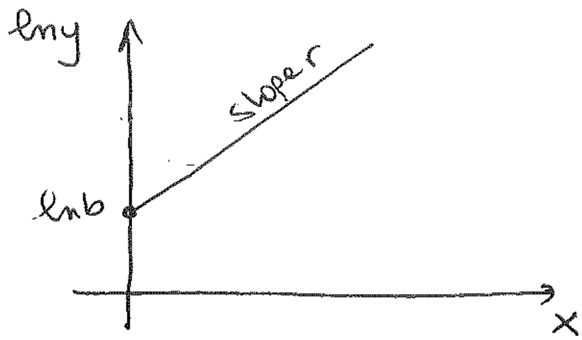
This is in fact generally true, and it is relatively easy to understand why this is the case.

Let us first look at the case of exponential functions.

Consider the graph of $y = b e^{rx}$

Now let's take the natural log of this:

$$\ln y = \ln(b a^x) = \ln b + \ln(e^{rx})$$
$$= \ln b + rx \ln e$$
$$\boxed{\ln y = \ln b + rx}$$



→ plotting $\ln y$ as a function of x is a straight line with slope r & y-intercept $\ln b$

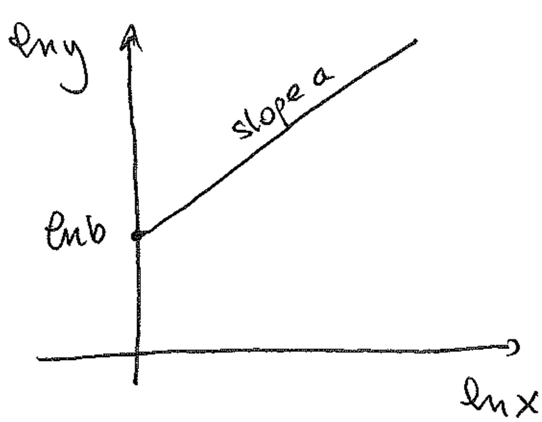
⇒ The log-linear axes <u>actually</u> plot $\ln y$ as a function of x (though usually label the y-axis with y).

<u>Next, the power laws</u>

Now consider $y = b x^a$ and take natural log:

$$\ln y = \ln(b x^a) = \ln b + a \ln x$$

→ if we plot $\ln y$ vs $\ln x$, we get a straight line with slope a & intercept $\ln b$!



⇒ The log-log axes <u>actually</u> plot $\ln y$ as a function of $\ln x$ (though usually label the axes with x & y).

Knowing that a logarithmic axis actually records the logarithm of a quantity (rather than the quantity itself) also explains why the numbers 0.1, 1, 10, 100, 1000 etc are equidistant on that axis. Indeed,

$\ln 1 = 0 \qquad \ln 10 = \text{some number}$

$\ln 100 = \ln(10^2) = 2\ln 10$

$\ln 1000 = \ln(10^3) = 3\ln 10$

all same distance

0 | ln10 | 2ln10 | 3ln10 | plot

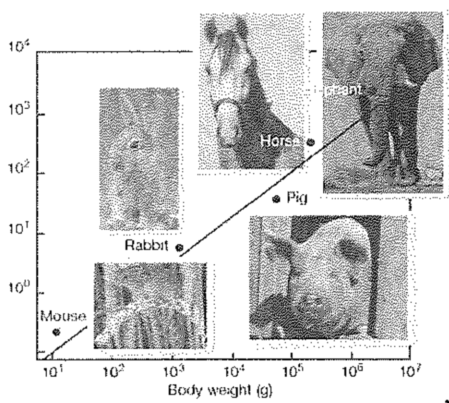
"1" | "10" | "100" | "1000" | label

To summarize, we have the following results:

- The logarithmic axis actually plots the logarithm of a quantity $\log x$ (or $\log y$), though often labels it as x (or y).
- The graph of an exponential function $f(x) = be^{\pm rx}$ appears as a straight line on a log-linear plot. The slope of the line is equal to $\pm r$.
- The graph of a power function $f(x) = bx^a$ appears as a straight line on a log-log plot. The slope of the line is the exponent a, and is positive if $a > 0$, and negative if $a < 0$.

Graphing on log-linear or log-log plots is used in a vast range of scientific papers. to *establish* that a given dataset is best modeled by a power law or an exponential. For instance:

log axis (relabelled)

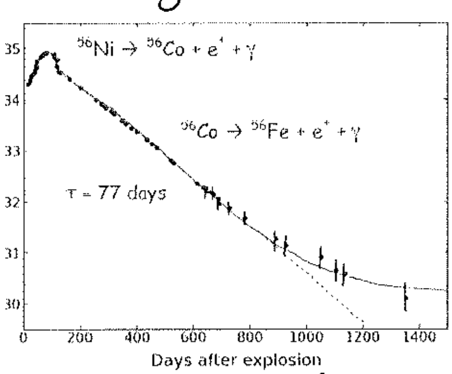


log axis (relabelled)

Kleiber's law of metabolism

→ a straight line on a log-log plot, it is therefore a power law

log axis (not relabelled)



(linear axis)

The light intensity of a supernova after explosion

→ a straight line on a log-linear plot → it is an exponential law (at least for some period of time).

5.5 The Gaussian function

One of the most important functions in many applied fields is the Gaussian function, also called Normal function or more commonly called the bell curve. The function is given by the following formula:

$$f(x) = a\, e^{-\frac{(x-m)^2}{2w^2}}$$ where a, m and w are real numbers

- a is the amplitude
- m is the mean (the average)
- w is the width (the standard deviation)

We will now see a Case Study that uses this function.

5.5.1 *Case Study: The distribution of IQ test scores*

IQ tests are standardized tests designed in the late 19th and early 20th century to assess a person's "intelligence". Modern IQ tests measure several factors which are related to intelligence, namely logical reasoning, math skills, language abilities, spatial relations skills, knowledge retained and the ability to solve novel problems.

The result of these tests is a single numeric score x. A distribution of scores can be created by recording how many people scored a particular value x, and dividing that value by the total number of test takers. This creates a function $N(x)$, which records the percentage of people whose score on the test was exactly x. The tests have been designed so that the distribution of test results (given a large enough sample of test-takers), is

$$N(x) = \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2(15)^2}}$$

Based on the definition above, this is a Gaussian function with

- $a = \frac{1}{15\sqrt{2\pi}}$
- $m = 100$
- $w = 15$

Using this function, we can now answer a few questions. For instance, what is the percentage of people who scored exactly 100 points (the most likely score)?

$$N(100) = \frac{1}{15\sqrt{2\pi}} \cdot e^{-\frac{(100-100)^2}{2(15)^2}} = \frac{1}{15\sqrt{2\pi}} e^{0} = \frac{1}{15\sqrt{2\pi}} \cong 0.0266$$
$$\simeq 2.66\%$$

To get into MENSA, your score on the IQ test must be above 130. What is the percentage of people whose score is exactly 130?

$$N(130) = \frac{1}{15\sqrt{2\pi}} e^{-\frac{(130-100)^2}{2(15)^2}} = \frac{1}{15\sqrt{2\pi}} e^{-\frac{30^2}{2(15)^2}} = \frac{1}{15\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{30}{15}\right)^2}$$

$$= \frac{1}{15\sqrt{2\pi}} e^{-\frac{1}{2}\cdot 4} = \frac{1}{15\sqrt{2\pi}} e^{-2} = 0.0036$$
$$= 0.36\%$$

What scores can you get with a probability of exactly 1 percent?

→ solve $N(x) = \frac{1}{100}$ for x

$$\frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2(15)^2}} = 0.01 \Rightarrow e^{-\frac{(x-100)^2}{2(15)^2}} = 0.01 \cdot 15 \cdot \sqrt{2\pi} = 0.15\sqrt{2\pi}$$

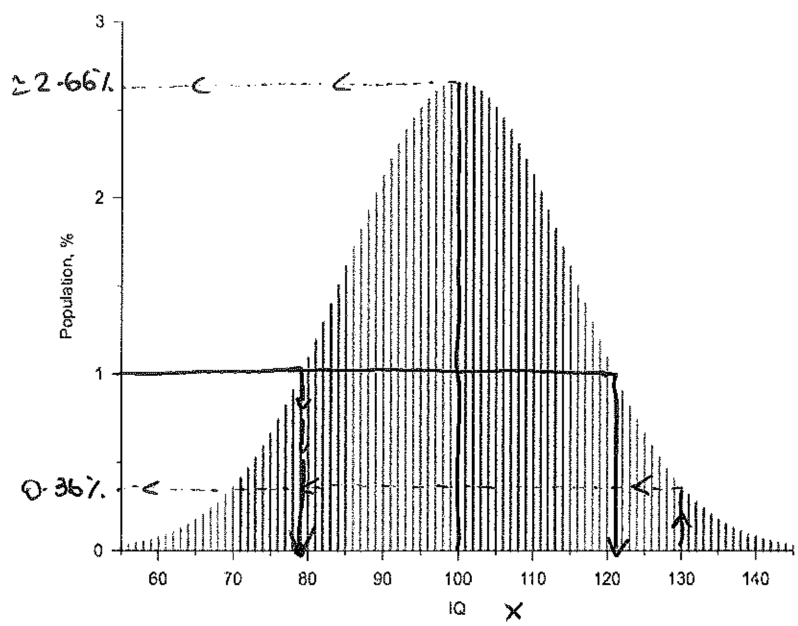
$$\text{so} \quad -\frac{(x-100)^2}{2(15)^2} = \ln(0.15\sqrt{2\pi})$$

$$\Rightarrow (x-100)^2 = -2(15)^2 \ln(0.15\sqrt{2\pi}) \simeq 440$$

$$\Rightarrow x-100 \simeq \pm\sqrt{440} \Rightarrow x \simeq 100 \pm \sqrt{440}$$

→ 2 possible solutions, $x_1 \simeq 100 + \sqrt{440} \simeq 121$
$x_2 \simeq 100 - \sqrt{440} \simeq 79$

We can now check our answers on the graph of the function $N(x)$, which has the well-known Bell-Curve shape:



Note how the graph of $N(x)$ looks like a "bell curve". This is true of all Gaussians.

We therefore see that while it is likely that someone will score a value close to 100, scores much lower than, say 70, or greater than, say, 130, are fairly unlikely. This is a defining standard property of the Gaussian function: it is a function that peaks at a certain value, and drops to 0 quite rapidly on either side of that value. Let us now see other properties of the Gaussian.

5.5.2 General properties of the Gaussian function

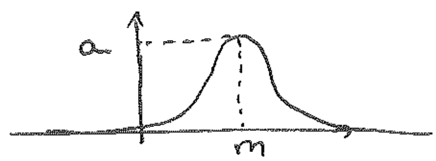
The Gaussian function has the following properties:

THE MAXIMUM OF THE GAUSSIAN. Given the expression $f(x) = a e^{-\frac{(x-m)^2}{2w^2}}$,

we see that $e^{-\frac{(x-m)^2}{2w^2}}$ is <u>maximum</u> when $-\frac{(x-m)^2}{2w^2}$ is <u>minimum</u>. The minimum of $-\frac{(x-m)^2}{2w^2}$ is at <u>$x = m$</u>

So the <u>maximum of the Gaussian is at $x = m$.</u>

The value of that maximum is $f(m) = a e^{-\frac{(m-m)^2}{2w^2}} = a e^0 = a$

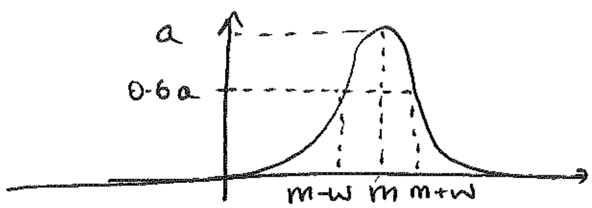


THE WIDTH OF THE GAUSSIAN.

The width of the Gaussian is w. To see this, note how

$$f(m+w) = a e^{-\frac{(m+w-m)^2}{2w^2}} = a e^{-\frac{w^2}{2w^2}} = a e^{-\frac{1}{2}} \cong 0.6a$$

$$f(m-w) = a e^{-\frac{(m-w-m)^2}{2w^2}} = a e^{-\frac{(-w)^2}{2w^2}} = a e^{-\frac{1}{2}} \simeq 0.6a$$



Based on this we see that different Gaussians can be obtained by changing the amplitude, mean and width of the Gaussian.