5.2 General logarithmic functions

Textbook Section 4.4

5.2.1 Case study: *The photic zone in the Ocean*

The photic zone is the upper layer of the ocean or a lake where there is enough light to support life, i.e. where phytoplanktons or algae can grow, and become part of the food chain for increasingly large predators. The intensity of sunlight in water decreases with depth under water, as it gets absorbed by water molecules. Because of this, very little light penetrates below a certain depth. The bottom of the photic zone is usually defined to be where the light intensity drops to about 1% of its surface value. How deep this is depends on the clarity of the water. In the clear open ocean, the light intensity drops by a factor of 2 roughly every 30 meters. In more murky lakes and ponds, the light intensity drops much more rapidly, by a factor of 2 roughly every 2 meters instead. Based on this information, how deep is it in the ocean, how deep is it in the lake? Answering this question will use what we learned from exponentials, and will introduce the need for logarithmic functions.

Let us begin by constructing a function to model the intensity of light as a function of depth below the surface, in meters, for the open ocean. For simplicity, let's assume that the intensity of light at the surface is exactly 1. (If you are worried about this assumption, note that we can always do this, by selecting a unit system in which the intensity of light at the surface is the unit intensity).

At depth 0: $I(d=0)=1$

At depth 30: $I(d=30)=\frac{1}{2}$

At depth 60: $I(d=60)=\frac{1}{4}=\frac{1}{2^2}$

At depth 90: $I(d=90)=\frac{1}{8}=\frac{1}{2^3}$

At depth 120: $I(d=120)=\frac{1}{16}=\frac{1}{2^4}$

$\vdots$

Clear pattern $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4} \dots$ that looks like an exponential → but how is the number 1, 2, 3, 4 ... related to depth? → it's $\frac{d}{30}$!

So finally

$$I(d)=\frac{1}{2^{\frac{d}{30}}}=2^{-\frac{d}{30}}$$

Based on this construction, let's also construct a function to model the intensity of light as a function of depth below the surface, in meters, for the lakes.

At depth 0: $I(0)=1$

At depth 2: $I(2)=\frac{1}{2}$

" " 4: $I(4)=\frac{1}{2^2}=\frac{1}{4}$

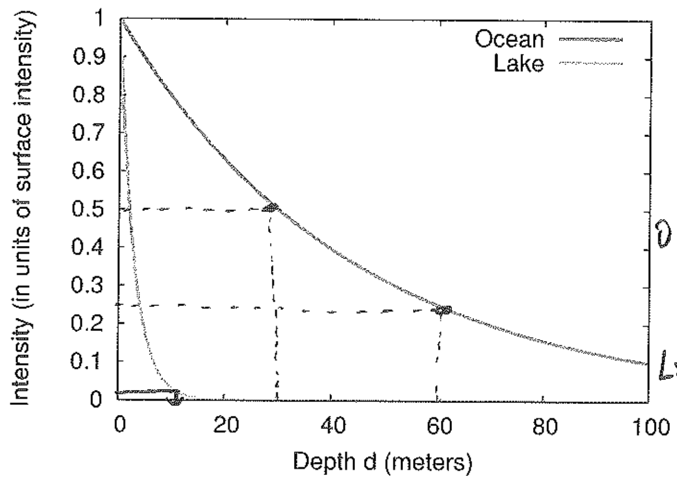
$\vdots$ 6: $I(6)=\frac{1}{2^3}=\frac{1}{8}$

$\vdots$

By similar argument we see that the exponent of 2 is $\frac{d}{2}$ so

$$I(d)=\frac{1}{2^{\frac{d}{2}}}=2^{-\frac{d}{2}}$$

The light intensity as a function of depth in both cases in shown in the following graph. We see that:



The light intensity drops much faster in lakes than in the ocean

In fact this comes from

$O: I(d) = 2^{-\frac{d}{30}} = \left(2^{\frac{1}{30}}\right)^{-d} = 1.023^{-d}$

while

$L: I(d) = 2^{-\frac{d}{2}} = \left(2^{\frac{1}{2}}\right)^{-d} = 1.41^{-d}$

→ the base in the ocean case is smaller than in the lake case

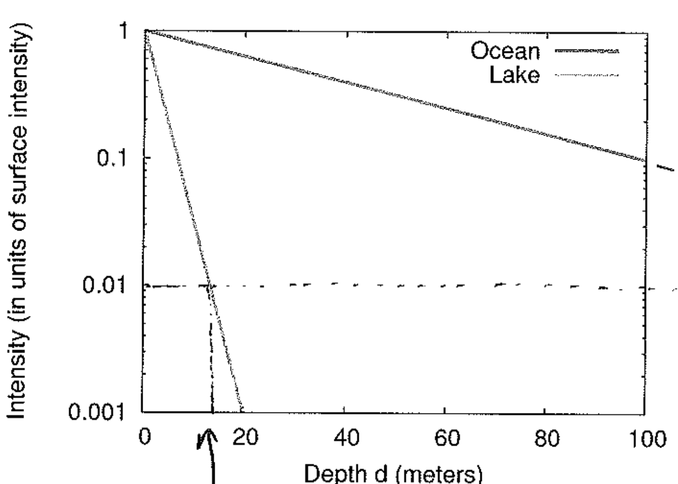
Let's try to use this graph to find where the bottom of the photic zone is:

→ we want to find where I is 1% of surface, so when $I = 0.01$

→ look on the y-axis where $I = 0.01$, & find which depth (on the x-axis) this corresponds to

For lakes, we see $d \simeq 11$ m, but for the ocean, it's hard to see

Unfortunately this graph has limitations, and is not as informative as we would like it to be. Here, we can use a trick that is similar to the one we used in the Case Study about the Rank-Size distribution of cities : logarithmic axes. To be precise, let's use a logarithmic axis for the y-axis only, and re-plot the data. We see that something remarkable happens (which will be explained next week):



Since we know it's a straight line we can extend it beyond the graph

$\simeq 200$ m.

more like 13 m

Thanks to this, we can get a better estimate the depth of the bottom of the photic zone for the lakes and the ocean:

Lakes: $d \simeq 13$ m Ocean: $d \simeq 200$ m

However, this method is still not very precise. Going back to the mathematical expression for the intensity of light, what mathematical equation would we have to solve in order to find the depth at which the light intensity drops to 1 percent of the surface intensity?

→ we need to solve $I = 0.01\, I_0$

Ocean case: $0.01 = 2^{-\frac{d}{30}}$

Lake case: $0.01 = 2^{-\frac{d}{2}}$

This kind of equation would be easy to solve if we knew what the inverse of an exponential function is... so let's find out what they are!

5.2.2 Definition and graph

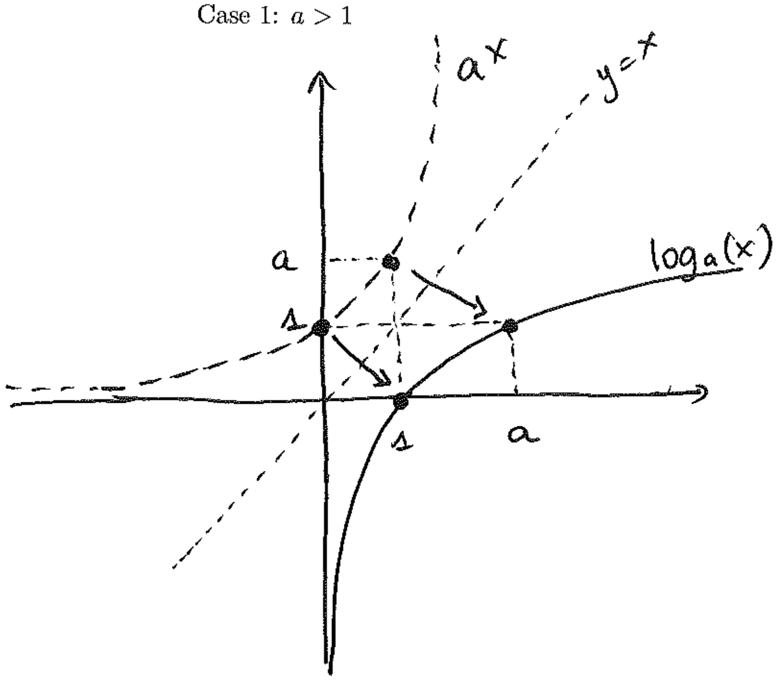
DEFINITION: The logarithm in base a, written as $\log_a(x)$ is the inverse of the exponential in base a: (a^x)

So if $y = a^x$ then $x = \log_a(y)$

Note that a <u>must</u> be > 0

GRAPH:

Case 1: $a > 1$



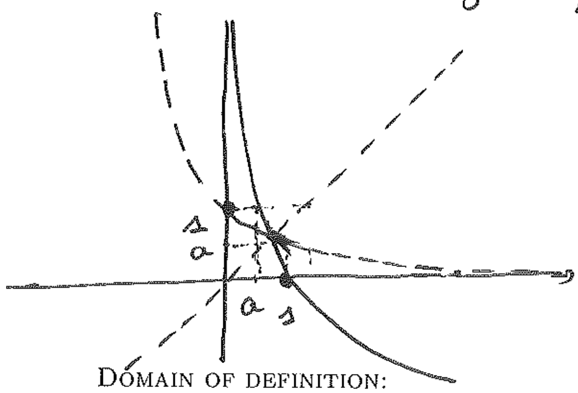
By this geometric construction we see that

- a^x has a horizontal asymptote at $x=0$

$\Rightarrow$ $\log_a(x)$ has a vertical asymptote at $y=0$

- $a^1 = a \Rightarrow \log_a(a) = 1$
- $a^0 = 1 \Rightarrow \log_a(1) = 0$

Case 2: $0 < a < 1$ (Rarely used) Write a^x as $\left(\frac{1}{A}\right)^{-x}$



again we see
- vertical asymptote at $y=0$
- $\log_a(1) = 0$
- $\log_a(a) = 1$

DOMAIN OF DEFINITION:

We see in all these graphs that $\log_a(x)$ is NOT DEFINED if $x \leq 0 \Rightarrow \mathcal{D} = (0, +\infty)$

UNIVERSAL PROPERTY OF LOGARITHMS: We also see from these graphs that
- $\log_a(1) = 0$
- $\log_a(a) = 1$

5.2.3 Examples of logarithms in common bases

THE FUNCTION $f(x) = \log_2(x)$ (LOGARITHM IN BASE 2)

x	2^x
-3	$1/8$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4
3	8

$\longrightarrow$ switch columns

x	$\log_2(x)$
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2

To find $\log_2(x)$ ask yourself the question "What is the power p such that $2^p = x$?"

THE FUNCTION $f(x) = \log_{10}(x)$ (LOGARITHM IN BASE 10)

x	10^x
-3	0.001
-2	0.01
-1	0.1
0	1
1	10
2	100
3	1000

switch columns $\longrightarrow$

x	$\log_{10}(x)$
0.001	-3
0.01	-2
0.1	-1
1	0
10	1
100	2
1000	3

To find $\log_{10}(x)$ ask yourself the question "What is the power p such that $10^p = x$"?

EXAMPLES:

- $\log_{10}(1000) =$ 1000 is 10^3 so $\log_{10}(1000) = 3$
- $\log_{10}(0.01) =$ 0.01 is 10^{-2} so $\log_{10}(0.01) = -2$
- $\log_2(0.25) =$ $0.25 = \frac{1}{4}$ is 2^{-2} so $\log_2(0.25) = -2$
- $\log_2(8) =$ $8 = 2^3$ so $\log_2(8) = 3$

5.2.4 The inverse relationships

Since the logarithm in base a is the inverse of the exponential in base a, we have the two fundamental relationships

with $f(x) = a^x$ $f^{-1}(x) = \log_a(x)$

- $f(f^{-1}(x)) = x \Rightarrow$ $\boxed{a^{\log_a(x)} = x}$
- $f^{-1}(f(x)) = x \Rightarrow$ $\boxed{\log_a(a^x) = x}$

These relationships can be used to solve exponential equations, such as the following examples:

EXAMPLES:

- Solve $3^x = 2$ apply $\log_3$ on both sides $\Rightarrow$

  $\log_3(3^x) = \log_3(2) \Rightarrow x = \log_3(2)$

- Solve $3^{-x} = 2$ apply $\log_3$ on both sides

  $\log_3(3^{-x}) = \log_3(2) \Rightarrow -x = \log_3(2)$

  $\Rightarrow x = -\log_3(2)$

- Solve $10^{-x} = -2$ apply $\log_{10}$ on both sides:

  $\log_{10}(10^{-x}) = \log_{10}(-2)$

  not allowed so there is no solution

- Solve $2^{-d/30} = 0.01$ for d apply $\log_2$ on both sides

  $\log_2\left(2^{-\frac{d}{30}}\right) = \log_2(0.01)$

  $-\frac{d}{30} = \log_2(0.01) \Rightarrow d = -30\log_2(0.01) = 199.3\text{m}$

- Solve $2^{-d/2} = 0.01$ for d same:

  $\log_2\left(2^{-\frac{d}{2}}\right) = \log_2(0.01) \rightarrow -\frac{d}{2} = \log_2(0.01)$

  $\rightarrow d = -2\log_2(0.01) = 13.28\text{m}$

These properties can also be used to solve logarithmic equations, or to simplify expressions with exponentials and logs...

EXAMPLES:

- $\log_2(2^x) =$ x
- $\log_5(5\sqrt{5}) =$ $\log_5\left(5 \cdot 5^{1/2}\right) = \log_5\left(5^{\frac{3}{2}}\right) = \frac{3}{2}$
- $\log_{10}(100^x) =$ $\log_{10}\left((10^2)^x\right) = \log_{10}\left(10^{2x}\right) = 2x$
- $3^{\log_3(2)} =$ 2
- $10^{\log_{100}(2x)} =$ $\left(100^{\frac{1}{2}}\right)^{\log_{100}(2x)} = \left(100^{\log_{100}(2x)}\right)^{\frac{1}{2}} = (2x)^{\frac{1}{2}} = \sqrt{2x}$
- Solve $\log_2(x) = 3$ $\rightarrow$ apply " 2^x " on both sides

  $2^{\log_2(x)} = 2^3 \rightarrow x = 9$
- Solve $2\log_{10}(x) = -4$ $\rightarrow$ apply " 10^x " on both sides

  $\log_{10}(x) = -\frac{4}{2} = -2 \Rightarrow 10^{\log_{10}(x)} = 10^{-2} = \boxed{0.01 = x}$
- Solve $2\log_{10}(-x) = -4$

  $\log_{10}(-x) = -\frac{4}{2} = -2 \Rightarrow 10^{\log_{10}(-x)} = 10^{-2} = 0.01 \Rightarrow -x = 0.01$

These relationships can also be used to prove important properties of logarithms... $\Rightarrow \boxed{x = -0.01}$

5.2.5 Properties of the logarithms and examples of use

Textbook Section 4.5

The following rules apply for logarithms.

- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a(x^b) = b\log_a(x)$
- $\log_a\left(\frac{1}{x}\right) = -\log_a(x)$
- $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$.

To show why these formulas are true, we go back to the definition of the logarithm as an inverse, and use the properties of the exponentials: for instance, to show why the first formula is true write:

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

raise as a^x on both sides

$$a^{\log_a(xy)} = a^{\log_a(x)+\log_a(y)} = a^{\log_a(x)} \cdot a^{\log_a(y)}$$

$$\underbrace{xy = x \cdot y}_{\text{obviously true}}$$

obviously true therefore original formula is true

Similarly, we can also show why the second formula is true:

$$\log_a(x^b) = b\log_a(x)$$

$$a^{\log_a(x^b)} = a^{b\log_a(x)}$$

$$x^b = \left(a^{\log_a(x)}\right)^b = x^b$$

obviously true therefore the original formula is true

Then, using this, we can now see why the other ones are true as well:

$$\log_a\left(\frac{1}{x}\right) = \log_a(x^{-1}) = -1\log_a(x) = -\log_a(x) \checkmark$$

$$\log_a\left(\frac{x}{y}\right) = \log_a\left(x \cdot \frac{1}{y}\right) = \log_a(x) + \log_a\left(\frac{1}{y}\right) = \log_a(x) - \log_a(y)$$

EXAMPLES:

- Combine into one log expression: $\log_2(x^2-1) - \log_2(x+1)$

$$\log_2(x^2-1) - \log_2(x+1) = \log_2\left(\frac{x^2-1}{x+1}\right) = \log_2\left(\frac{\cancel{(x+1)}(x-1)}{\cancel{x+1}}\right)$$
$$= \log_2(x-1)$$

- Simplify $\log_2(8(x-2))$:

$$\log_2(8(x-2)) = \log_2(8) + \log_2(x-2)$$
$$= 3 + \log_2(x-2)$$

- Simplify $\log_{10}(100^{x+1}) + \log_{10}\left(\frac{1}{5^x}\right)$

$$\log_{10}(100^{x+1}) + \log_{10}\left(\frac{1}{5^x}\right)$$
$$= (x+1)\log_{10}(100) - \log_{10}(5^x)$$
$$= (x+1) \cdot 2 - x\log_{10}(5) = 2x + 2 - x\log_{10}(5)$$

- Write the following expression as a sum or difference of logs: $\log_{10}\left[\frac{x-1}{(x+2)^2}\right]$

$$\log_{10}\left[\frac{x-1}{(x+2)^2}\right] = \log_{10}(x-1) - \log_{10}[(x+2)^2]$$
$$= \log_{10}(x-1) - 2\log_{10}(x+2)$$

- Write the following expression as a sum or difference of logs: $\log_3\left[\frac{3^{x+1}(x-3)^2(x+4)}{(x-2)^3 9^x}\right]$

$$\log_3[3^{x+1}(x+4)(x-3)^2] - \log_3[(x-2)^3 9^x]$$
$$= \log_3(3^{x+1}) + \log_3(x+4) + \log_3((x-3)^2)$$
$$- [\log_3((x-2)^3) + \log_3(9^x)]$$
$$= (x+1)\log_3(3) + \log_3(x+4) + 2\log_3(x-3)$$
$$- 3\log_3(x-2) - x\log_3(9)$$
$$= (x+1) + \log_3(x+4) + 2\log_3(x-3) - 3\log_3(x-2) - 2x$$
$$= 1 - x + \log_3(x+4) + 2\log_3(x-3) - 3\log_3(x-2)$$