Chapter 5

Exponentials and logarithms

5.1 General Exponential functions

Textbook Chapter 4.3

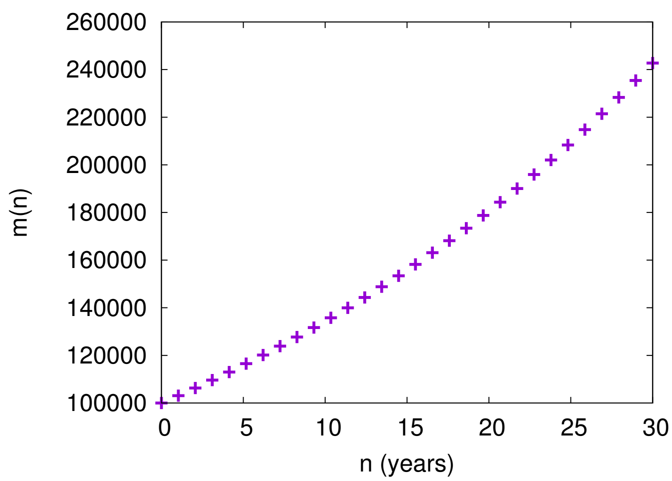
5.1.1 Case Study: *Compound interests*

When opening a savings account, a bank usually offers an interest rate compounded yearly. Suppose for simplicity that the interest rate is 3%. Compounding yearly means that the interests, at the rate of 3% of the total in your account, are calculated and added to your account once a year. This case study focuses on figuring out how money accrues in your account, starting from a total amount of money of $100,000.

Starting with $100,000 how much money will be in the account after 1 year? after 2 years? (assume that no money is taken out in between).

Based on this, how much money will be in the account after n years (assuming no money is ever taken out of the account)?

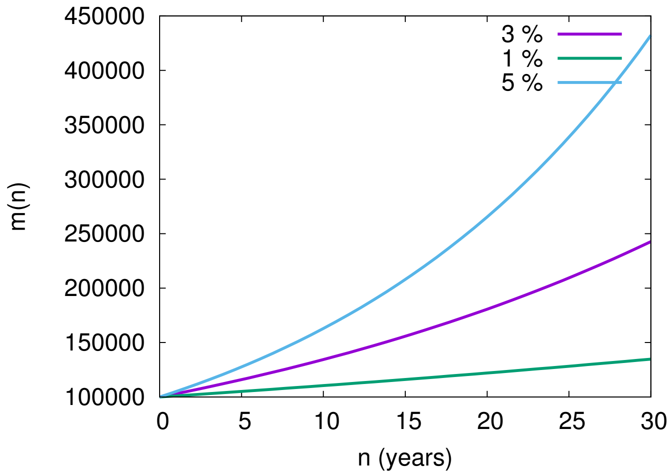
The following graph shows the function $m(n)$. We see that:



Suppose we now have a more realistic case scenario where the interest rate is 1% instead of 3%. What is the new function that describes the amount of money in your account as a function of year?

Suppose instead we consider an investment account with an estimated growth rate of 5% (and we are lucky enough that the stock market does not crash). What is the new function $m(n)$ now?

The following graph compares the functions $m(n)$ for the 3 different interest rates. We see that:



The functions that we have constructed are all exponential functions. Exponential functions play a crucial role in nearly every aspect of mathematical modeling, in ecology, epidemiology, physics, chemistry, economics (as we just saw) etc... We will now learn about some generic properties of exponentials.

5.1.2 Definition of an exponential functions

DEFINITION:

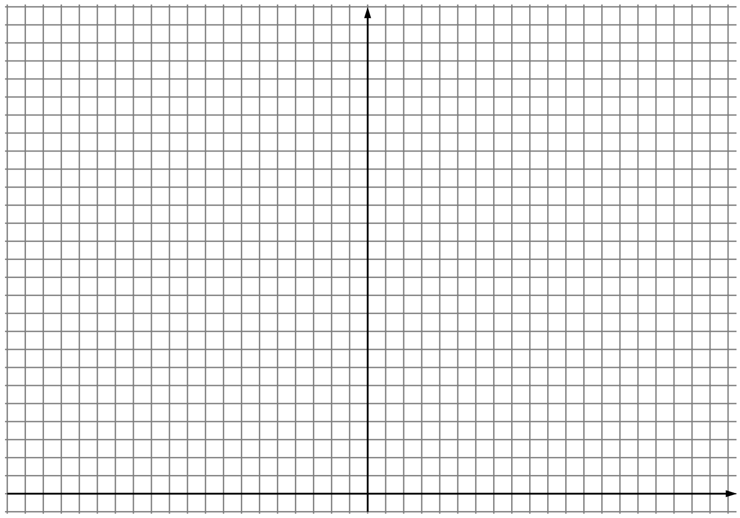
NOTE: Do not mix up power and exponential functions!

- For power functions:

- For exponential functions:

5.1.3 Graphs of exponential functions

While we may not be used to thinking of exponents as non-integers, or non-rational numbers, just construct the following tables for the functions $f(x) = 2^x$ and $g(x) = 2^{-x}$:



More generally, the graph of a basic exponential function $f(x) = a^x$ or $g(x) = a^{-x}$ depends on the value of the base a.

Case 1: $a > 1$ (typical example: $f(x) = 2^x$, or $g(x) = 2^{-x}$)

Case 2: $0 < a < 1$ (typical example: $f(x) = 0.5^x$ or $g(x) = 0.5^{-x}$)

Finally, knowing the graphs of basic exponential functions, we can now graph functions that are based on the latter, through simple geometric transformations. Here are some examples:

- Graph the function $f(x) = 3 + 2^{-x}$

- Graph the function $f(x) = 3^{x+1} - 1$

- Graph the function $f(x) = 1 - \left(\frac{1}{2}\right)^x$

5.1.4 Properties of exponential functions

Manipulation of exponential functions: The rules for manipulating these functions are the same as the rules for manipulating exponents. Given an exponential function in base a

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Also, given another exponential function in base b

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etc...

Examples:

- Simplify: $f(x) = \frac{3^{x+2}}{9}$
- Simplify $f(x) = \frac{2^{2x}}{4^x}$
- Simplify $f(x) = 25^x 5^{-x-1}$
- Simplify $f(x) = 2^{2x} 3^x$