Chapter 5

Exponentials and logarithms

5.1 General Exponential functions

Textbook Chapter 4.3

5.1.1 Case Study: *Compound interests*

When opening a savings account, a bank usually offers an interest rate compounded yearly. Suppose for simplicity that the interest rate is 3%. Compounding yearly means that the interests, at the rate of 3% of the total in your account, are calculated and added to your account once a year. This case study focuses on figuring out how money accrues in your account, starting from a total amount of money of $100,000.

Starting with $100,000 how much money will be in the account after 1 year? after 2 years? (assume that no money is taken out in between).

- At year $n=0 \to \$100{,}000$
- At year $n=1 \to \$100{,}000 + 3$ percent of $100{,}000$

$$= \$100{,}000 + \frac{3}{100}100{,}000 = \$103{,}000$$

Note that this is also the same as saying

$$= (1.03) \times 100{,}000$$

↑ add 3% is equivalent to multiplying by 1.03

- At year $n=2 \to \$103{,}000 + \frac{3}{100}103{,}000$

$$= \left(1+\frac{3}{100}\right) \times 103{,}000 = 1.03 \times 103{,}000$$

↖ add 3% ↔ multiply by 1.03

$$= (1.03) \times (1.03) \times 100{,}000$$

$$= (1.03)^2 \times 100{,}000 = \$106{,}090$$

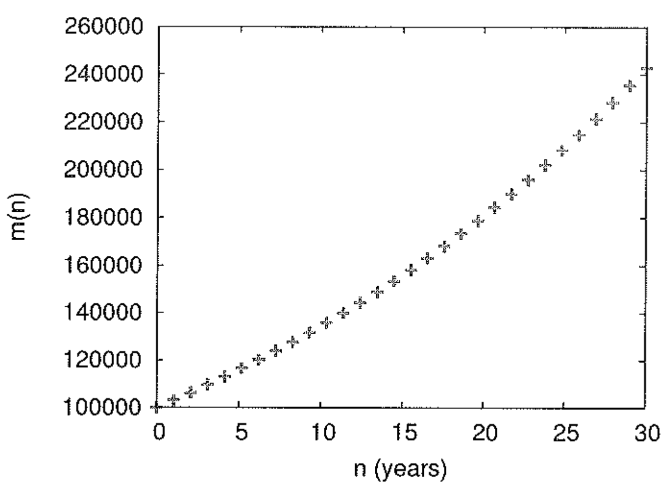
Based on this, how much money will be in the account after n years (assuming no money is ever taken out of the account)?

Each time we add 3% (every year) we multiply the value at the previous year by 1.03

→ $n=0$: 100,000
$n=1$: $100\,000 \times 1.03$
$n=2$: $100\,000 \times 1.03^2$
$n=3$: $100{,}000 \times 1.03^3$
$n=4$: $100{,}000 \times 1.03^4$
⋮

→ at nth year: $m(n) = 100{,}000 \times 1.03^n$

The following graph shows the function $m(n)$. We see that:



- The amount of money $m(n)$ as function of year is an increasing function
- The increase is faster than a linear increase, and accelerates over time (the curve gets steeper with increasing n)

Suppose we now have a more realistic case scenario where the interest rate is 1% instead of 3%. What is the new function that describes the amount of money in your account as a function of year?

This time "adding $\frac{1}{100}$" each year is equivalent to multiplying by 1.01 each year:

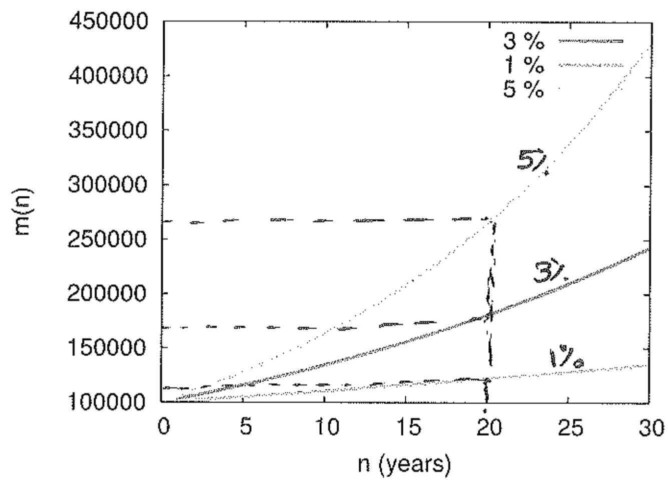
$n=0$: 100,000
$n=1$: $100{,}000 \times 1.01$
$n=2$: $100{,}000 \times 1.01^2$
⋮
n : $\boxed{100{,}000 \times 1.01^n}$

Suppose instead we consider an investment account with an estimated growth rate of 5% (and we are lucky enough that the stock market does not crash). What is the new function $m(n)$ now?

→ this time we multiply by 1.05 each year

so $m(n) = 100{,}000 \times (1.05)^n$

The following graph compares the functions $m(n)$ for the 3 different interest rates. We see that:



The rate of increase of money in the account is <u>much higher</u> for a 5% rate than a 1% rate. (more than a factor of 5!)
After 20 years for instance
- at 1% $m(20) = 100{,}000\,(1.01)^{20} \simeq 122{,}000$
- at 5% $m(20) = 100{,}000\,(1.05)^{20} = 265{,}330$

The functions that we have constructed are all exponential functions. Exponential functions play a crucial role in nearly every aspect of mathematical modeling, in ecology, epidemiology, physics, chemistry, economics (as we just saw) etc... We will now learn about some generic properties of exponentials.

5.1.2 Definition of an exponential functions

DEFINITION: An exponential function is any function of the kind $f(x) = b.(a^x)$ or $f(x) = b.\, a^{-x}$ where b and a are constant. <u>a is called the base</u>

NOTE: Do not mix up power and exponential functions!

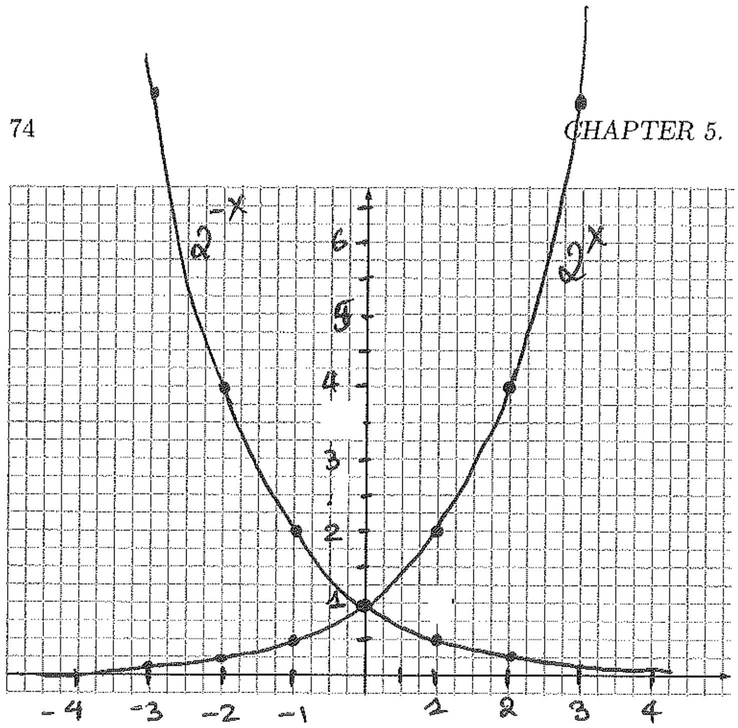
- For power functions: $f(x) = bx^a$
- For exponential functions: $f(x) = ba^x$

→ For exponentials x is in exponent

5.1.3 Graphs of exponential functions

While we may not be used to thinking of exponents as non-integers, or non-rational numbers, just construct the following tables for the functions $f(x) = 2^x$ and $g(x) = 2^{-x}$:

	-3	-2	-1	0	1	2	3
2^x	1/8	1/4	1/2	1	2	4	8
2^{-x}	8	4	2	1	1/2	1/4	1/8

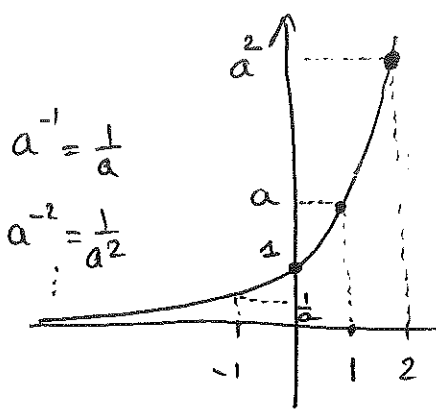


The exponential 2^x grows very rapidly when $x \to \infty$ and tends to the x-axis rapidly when $x \to -\infty$

The exponential 2^{-x} is the mirror image of 2^x with respect to the y-axis. It goes to 0 (to the x-axis) quickly when $x \to \infty$, and to $+\infty$ when $x \to -\infty$

More generally, the graph of a basic exponential function $f(x) = a^x$ or $g(x) = a^{-x}$ depends on the value of the base a.

Case 1: $a > 1$ (typical example: $f(x) = 2^x$, or $g(x) = 2^{-x}$)



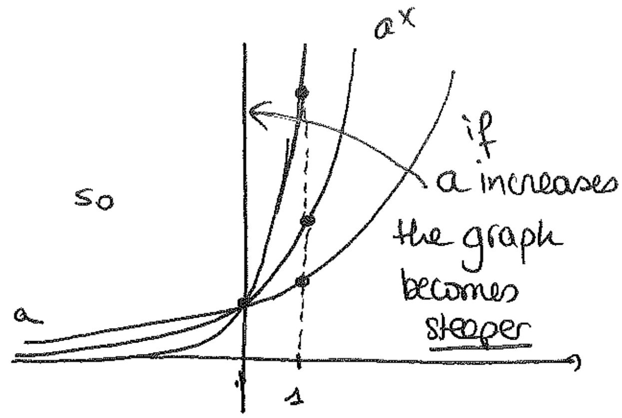
Recall $a^0 = 1$

$a^1 = a$

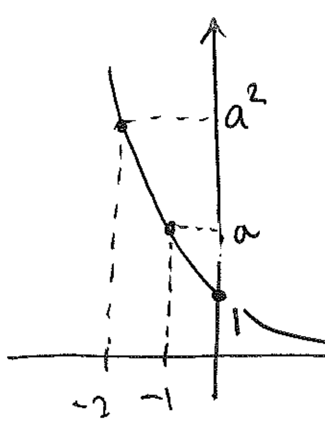
$a^2 = a^2$

→ when you add 1 to x, you multiply f(x) by a

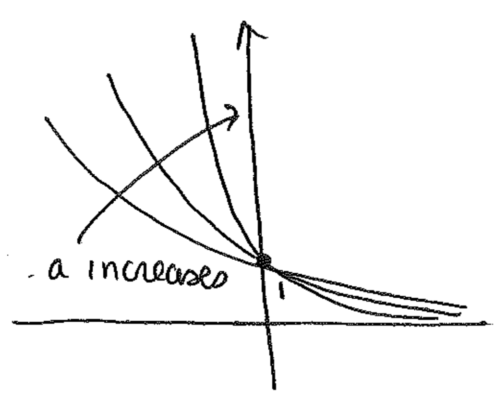
so



The functions a^{-x} are the mirror image of these so



when you add 1 to x, you divide f(x) by a

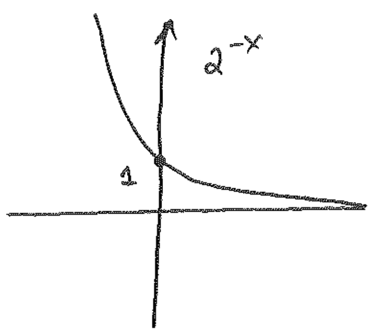


Case 2: $0 < a < 1$ (typical example: $f(x) = 0.5^x$ or $g(x) = 0.5^{-x}$)

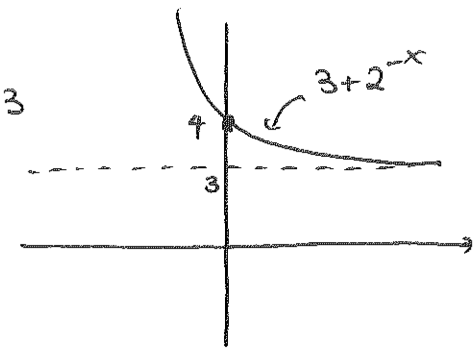
- Remember that if $0 < a < 1$, you can write a as $\frac{1}{A}$ where $A > 1$ (as in $\underset{a}{0.5} = \frac{1}{\underset{A}{2}}$)
- Then write $a^x = \left(\frac{1}{A}\right)^x = A^x$
  $a^{-x} = \left(\frac{1}{A}\right)^{-x} = A^{-x}$] and use the graphs in the $A > 1$ case above

Finally, knowing the graphs of basic exponential functions, we can now graph functions that are based on the latter, through simple geometric transformations. Here are some examples:

- Graph the function $f(x) = 3 + 2^{-x}$

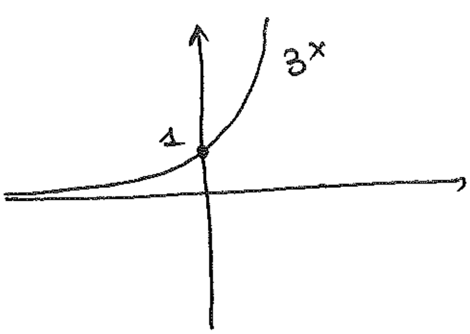


raise by 3 $\Rightarrow$

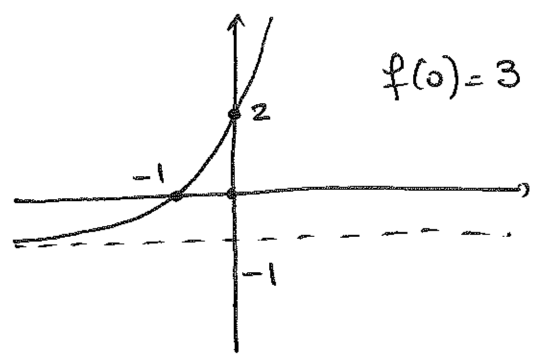


$f(0) = 3 + 2^0$
$= 3 + 1$
$= 4$

- Graph the function $f(x) = 3^{x+1} - 1$ $\quad 3^{x+1} - 1 \rightarrow$ shift to left by 1, down by 1

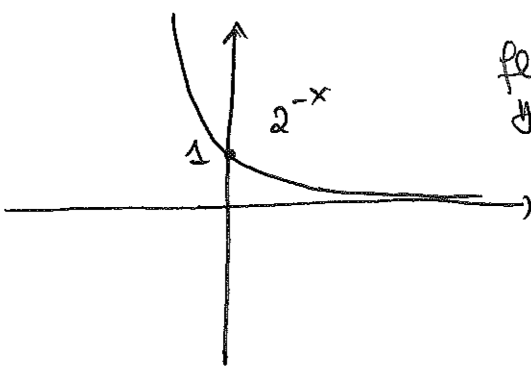


$\longrightarrow$

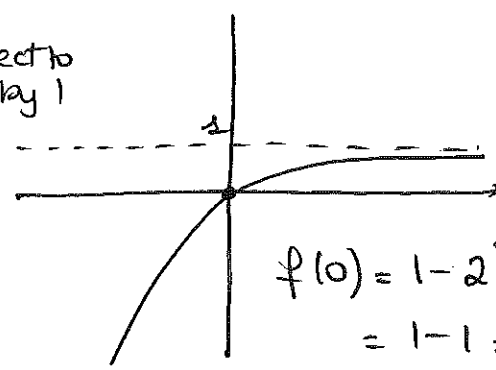


$f(0) = 3^1 - 1 = 2$

- Graph the function $f(x) = 1 - \left(\frac{1}{2}\right)^x$ $\quad \left(\frac{1}{2}\right)^x = 2^{-x}$ so $f(x) = 1 - 2^{-x}$



flip with respect to y axis, raise by 1 $\longrightarrow$



$f(0) = 1 - 2^0$
$= 1 - 1 = 0$

5.1.4 Properties of exponential functions

MANIPULATION OF EXPONENTIAL FUNCTIONS: The rules for manipulating these functions are the same as the rules for manipulating exponents. Given an exponential function in base a

- $a^0 = 1$
- $a^1 = a$
- $a^{x+y} = a^x a^y$
- $a^{-x} = \frac{1}{a^x}$
- $\frac{a^x}{a^y} = a^{x-y}$
- $(a^x)^y = a^{xy} = (a^y)^x$

Also, given another exponential function in base b

- $a^x b^x = (ab)^x$
- $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$
- $a^x b^{-x} = a^x \left(\frac{1}{b}\right)^x = \left(\frac{a}{b}\right)^x$

etc...

EXAMPLES:

- Simplify: $f(x) = \frac{3^{x+2}}{9}$

$$\frac{3^{x+2}}{9} = \frac{3^x 3^2}{9} = \frac{3^x \cdot \cancel{9}}{\cancel{9}} = 3^x$$

- Simplify $f(x) = \frac{2^{2x}}{4^x}$

$$\frac{2^{2x}}{4^x} = \frac{(2^2)^x}{4^x} = \frac{4^x}{4^x} = 1$$

- Simplify $f(x) = 25^x 5^{-x-1}$

$$25^x 5^{-x-1} = (5^2)^x 5^{-x-1} = 5^{2x} 5^{-x-1} = 5^{2x-x-1} = 5^{x-1}$$

- Simplify $f(x) = 2^{2x}3^x$

$$2^{2x} 3^x = (2^2)^x 3^x = 4^x 3^x = (4 \times 3)^x = 12^x$$