Chapter 4

Powers functions. Composition and Inverse of functions.

4.1 Power functions

We have already encountered some examples of power functions in the previous chapters, in the context of polynomial and rational functions: indeed, functions such as

$$f(x) = 3x^2 \qquad f(x) = \frac{7}{x^4} \qquad f(x) = x^{-6} \qquad f(x) = 2x^8$$

as power functions. More generally speaking, power functions are defined as follows:

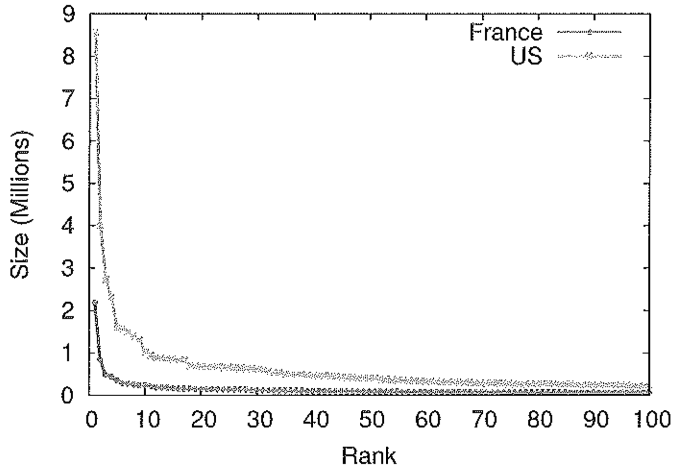
DEFINITION:
- A power function is any function of the kind $f(x) = ax^b$ where a and b are any real number
- Unless b is integer, only $x \geq 0$ is allowed.

EXAMPLES:
- the functions above
- $f(x) = 3x^{2/5}$, $f(x) = 4x^{-7/3}$, $f(x) = x^0$, ...
- $f(x) = \frac{1}{2}x^{\pi}$, $f(x) = \sqrt{2}\,x^{-\sqrt{3}}$, etc...

Power functions are fairly ubiquitous in natural systems, and in many examples, the power is not an integer. We will now work through a case study that showcases power functions in the context of socioeconomics.

4.1.1 Case Study: *The Rank-Size law of cities*

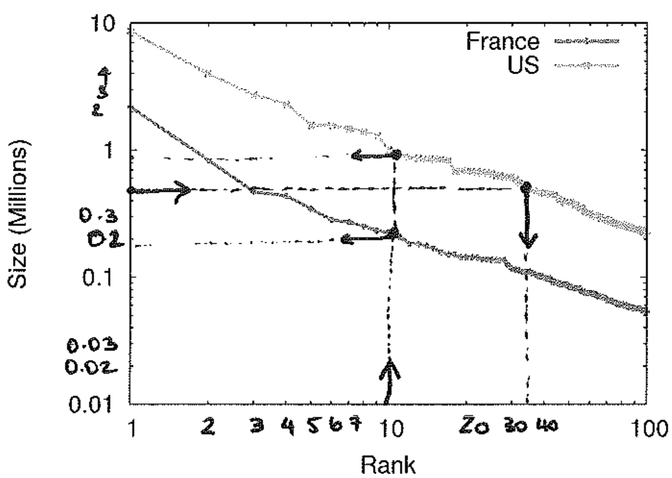
A remarkable empirical law of socio-economics is the observed relationship between the rank and size of the population of each city in a given country. To construct a graph showing this relationship, simply graph the population of a city against its rank (i.e. 1 for largest, 2 for second largest, 3 for third largest, etc..). Here are two examples, one for the US, and one for France:



we see that

- The size decreases with increasing rank (as expected from definition), and looks a little bit like $\frac{1}{x}$ or $\frac{1}{x^2}$ function
- For given rank, US city size is always larger.

Unfortunately, the graph is not as informative as we would like, mostly because the population of the largest cities is so much larger than that of the smaller ones. An alternative way of plotting the data is to use a log-log scale. We will learn more about them in a few lecture's time, but just note for now how, on each axis, the numbers 1, 10, 100, etc. are equally-spaced, instead of the numbers 1,2,3,... being equally spaced. When using a log-log plot, something remarkable happens to this data:



- The data lies nearly on a straight line!
- The line for US cities is nearly parallel to the line for french cities (and lies above, as expected).

As we will learn shortly, when data falls on a straight line in a log-log plot, this is symptomatic of a power-law relationship. It is in fact how scientists prove that a relationship is a power law. Let's try to fit the data with a function of the kind $s = f(r) = ar^b$, where s is the size, and r is the rank: we get

US: $S = 6\, 10^6\, r^{-0.7}$

France: $S = 1.3\, 10^6\, r^{-0.7}$

Using these two functions, we therefore see that the 10th largest cities in France and the US have population size of about

US: $S = 6\,10^6 \cdot 20^{-0.7} = 1.2$ Million (Reality: San Jose, 1 Million)

France: $S = 1.3\,10^6 \cdot 20^{-0.7} = 260{,}000$ (Reality: Lille, 200,000)

→ The functions overestimate the actual size by a little bit, but not much.

We will now learn a little more about power laws, both in terms of where they occur, and what we can do with them.

4.1.2 More examples of power laws in nature

There are many examples of power laws in nature.

- Some of the most common ones are geometric relationships such as :

  Volume of sphere of radius r: $V(r) = \frac{4}{3}\pi r^3$

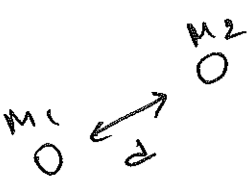
  Area of circle of radius r: $A(r) = \pi r^2$

  Circumference of circle of radius r: $C(r) = 2\pi r$

- Another common class of power laws are allometric laws in biology:

  - Kleiber's Law for metabolic rate as function of mass
    $\text{Rate} = 70\,(\text{Mass})^{0.75}$ Mass in kg, Rate in kcal/day
  - Optimal flight speed for bird/plane as function of mass:
    $\text{Speed} = 30\,\text{Mass}^{1/6}$ (Mass in kg, speed in m/s)

- In fact, there are examples of power laws in most fields of Science!



  - Force of gravitation: $\text{Force} = \frac{G M_1 M_2}{d^2}$ distance d
    (Newton's Law)
  - Gutenberg - Richter Law:
    Number of Earthquakes $\simeq a \cdot \text{Magnitude}^{-1}$
    a number depending on time period & area covered.

4.1.3 Manipulations of power functions

The following rules of exponents apply for manipulating power functions:

- $x^0 = 1$ for any x
- $x^1 = 1$ for any x
- $x^a x^b = x^{a+b}$ for any x, a, b
- $x^{-a} = \frac{1}{x^a}$ for any x, a
- $\frac{x^a}{x^b} = x^a x^{-b} = x^{a-b}$ for any x, a, b
- $(x^a)^b = x^{ab} = (x^b)^a$ for any x, a, b

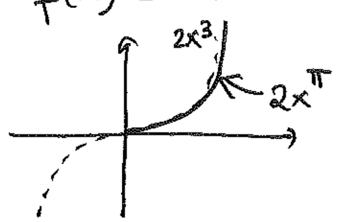
4.1.4 Graphs of power functions

The overall shape of the graph of a power function depends on the sign and value of the exponent b, as well as the number a of in front of course.

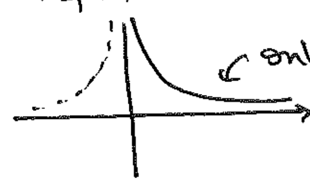
- For $b < -1$ or $b > 1$, the graph looks like the integer power function with nearest power, e.g

$f(x) = 2x^{\pi}$

π closest to 3:
$f(x)$ close to $2x^3$
(but only $x \geq 0$)

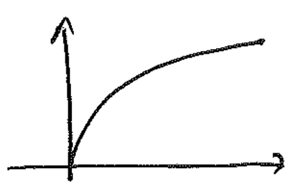


$f(x) = 3x^{-\sqrt{6}}$ $\quad \sqrt{6} \simeq 2.4$
$\rightarrow f(x)$ close to $3x^{-2}$



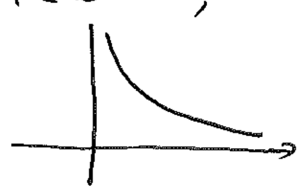
only $x > 0$ part.

- for $0 < b < 1$, the graph looks like $\sqrt{x}$:



$\rightarrow f(x) = x^{1/3}$, $x^{\frac{1}{\pi}}$ all look like this
(but only $x \geq 0$ part)

- For $-1 < b < 0$, the graph looks like $\frac{1}{x}$



(but only $x > 0$ part).

4.2 Inverse of functions and Composition of functions

Textbook section 4.2

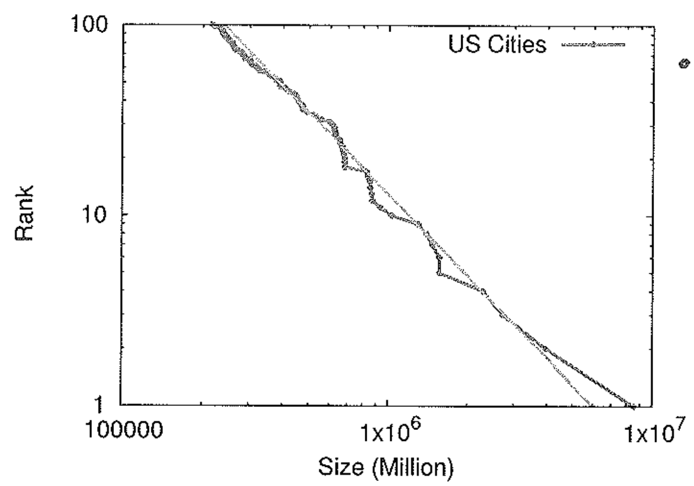
4.2.1 Case Study: *The Rank-Size law of cities*

Sacramento is the capital of California, and has a population size of about 490,000 people. What is its rank based on the graph?

Start from 490,000 on y-axis, then look up x-axis value
→ we find $r \simeq 35$

This is in fact a fairly good estimate, as the true rank of Sacramento is exactly 35.

We can apply the same technique to many cities on this graph, i.e. look up their rank based on their size. We could in fact create a graph with this information, which would allow us to retrieve it and share it with others much more efficiently. This would give the following plot (on a log-log scale):



- This is equivalent to just swapping the x- and y axes!
- We see that the data also lies on a straight line of course

Now, what about Santa Cruz? The population of Santa Cruz is roughly 70,000 people, so what is its rank? Unfortunately, the graph above is not very useful because it does not show what happens for cities of less than 200,000 people. So what can we do? The solution here is not to use the graph, but to use what we know about the Rank-Size relationship of US cities:

From $s = 6\,10^6\, r^{-0.7} \to 70{,}000 = 6\,10^6\, r^{-0.7}$, solve for r:

$$\frac{70{,}000}{6\,10^6} = r^{-0.7} \to \left(\frac{70{,}000}{6\,10^6}\right)^{\frac{1}{-0.7}} = \left(r^{-0.7}\right)^{\frac{1}{-0.7}} = r^1$$

$$\to r = \left(\frac{70000}{6000000}\right)^{-1/0.7} = 577 \text{ th city in US.}$$

In fact, we could do this for any US city, namely, to get an estimate of the rank as a function of their size:

$$s = 6\,10^6\, r^{-0.7} \to \frac{s}{6\,10^6} = r^{-0.7}$$

$$\to \left(\frac{s}{6\,10^6}\right)^{\frac{1}{-0.7}} = \left(r^{-0.7}\right)^{\frac{1}{-0.7}} = r^1 = r$$

$$\Rightarrow \boxed{r = \left(\frac{s}{6\,10^6}\right)^{-\frac{1}{0.7}} = g(s)}$$

In doing so, we have created another function $g(s)$, which takes the size and returns the rank: $r = g(s)$. If we plot this new function on the graph of Rank vs. Size, we see that it indeed fits the data very well. We can therefore use this, as an alternative option to the graph, to find the estimated rank of any city in the US knowing its size!

The functions f and g are clearly related to one another:

$$\text{rank} \underset{g}{\overset{f}{\rightleftarrows}} \text{size}$$

f takes rank, gives size

g takes size, gives rank

In fact, they are called Inverse of on another! We will now learn more about inverses, and generalize the concept we have just learned.

4.2.2 Definition of the inverse, and examples

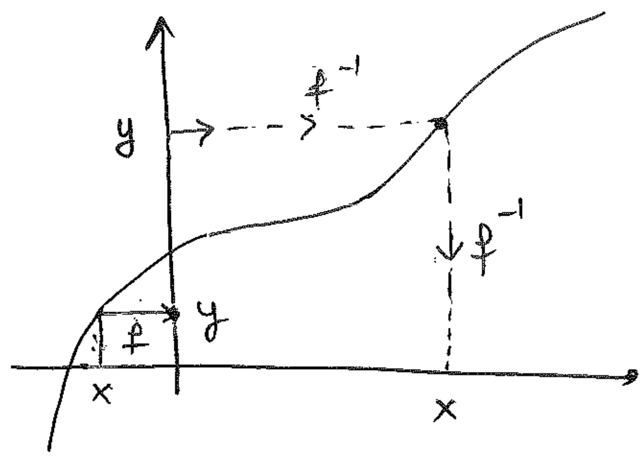
DEFINITION: The inverse of a function $f(x)$ is another function, usually written $f^{-1}(x)$, which <u>reverses</u> the action of f

$$x \underset{f^{-1}}{\overset{f}{\rightleftarrows}} y$$

$$\boxed{\begin{array}{l} y = f(x) \text{ takes } x\text{, gives } y \\ x = f^{-1}(y) \text{ takes } y\text{, gives } x \end{array}}$$

GRAPHICAL INTERPRETATION:

The inverse function answers the question "If I know y, which x did it come from?"



Finding the function f^{-1} given $y = f(x)$ boils down to taking $y = f(x)$, <u>knowing y and solving for x</u>.

IMPORTANT NOTES:

- $f^{-1}(x) \neq \frac{1}{f(x)}$! f^{-1} is just a notation
- If f^{-1} is the inverse of f, then f is the inverse of f^{-1}
- It is customary to switch x & y at the end to write $f^{-1}(x) = \dots$

EXAMPLES

- $y = f(x) = 3x + 2$: $y = 3x+2 \to$ solve for x

  $y-2 = 3x \Rightarrow x = \frac{y-2}{3} = f^{-1}(y)$

  $\to$ ∴ to know which x a given y came from, simply calculate $x = f^{-1}(y) = \frac{y-2}{3}$!

  ↑ this is the inverse of $f(x) = 3x+2$
- $y = f(x) = x^2$ (for $x \geq 0$):

  $y = x^2 \to x = \sqrt{y} = f^{-1}(y)$

  $\to$ given y, to know which x it came from just calculate $x = f^{-1}(y) = \sqrt{y}$ ← this is the inverse of $f(x) = x^2$
- $y = f(x) = \sqrt{x-2}$ (for $x \geq 2$):

  $y = \sqrt{x-2} \Rightarrow y^2 = x+2 \Rightarrow x = y^2 - 2 = f^{-1}(y)$

  $\Rightarrow f^{-1}(y) = y^2 - 2$ is the inverse of $f(x) = \sqrt{x-2}$
- $y = f(x) = \frac{3x+1}{x-2}$

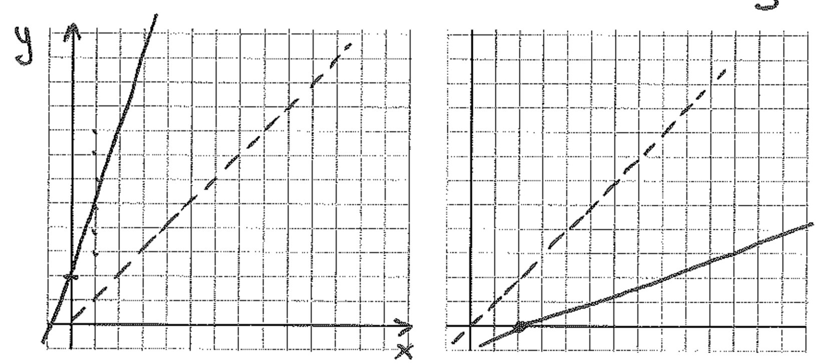
  $y = \frac{3x+1}{x-2} \to (x-2)y = 3x+1 \to xy - 2y = 3x+1$

  $\to xy - 3x = 2y+1 \to x(y-3) = 2y+1$

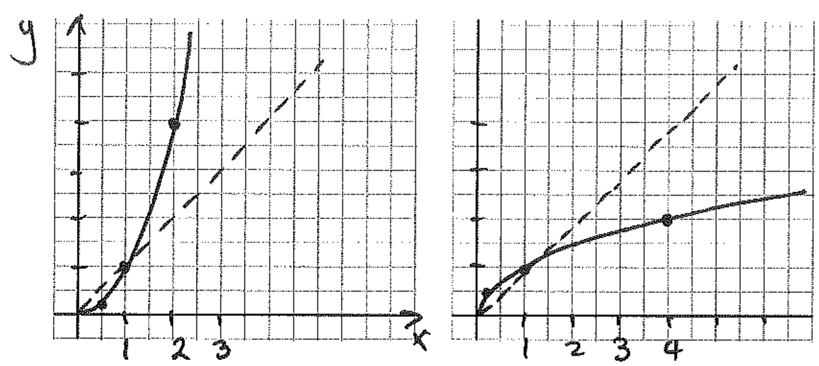
  $\to x = \frac{2y+1}{y-3} \to$ this is the inverse of $f(x) = \frac{3x+1}{x-2}$.

4.2.3 Graph of an inverse function and horizontal line test:

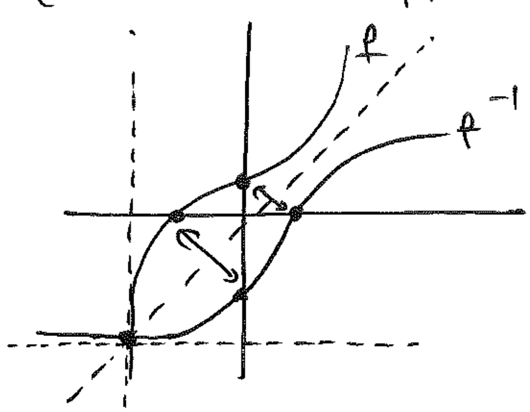
EXAMPLE 1: $y = f(x) = 3x + 2$ $\qquad f^{-1}(x) = \frac{x-2}{3}$



EXAMPLE 2: $y = f(x) = x^2$ $\qquad f^{-1}(x) = \sqrt{x}$



So from these graphs we notice that: the graph of $f^{-1}(x)$ is the mirror-image of the graph of $f(x)$ with respect to the $y=x$ line (recall we are effectively switching the x & y axes!)



- x-intercepts become y-intercepts & vice versa
- points on the $y=x$ line stay where they are
- horizontal asymptotes become vertical ones, & vice versa

NOTE: It may happen that through this process, the graph of the inverse does not satisfy the vertical line test: in that case, the inverse is not defined.

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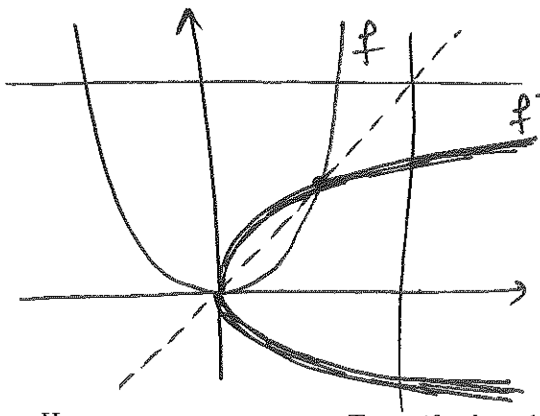
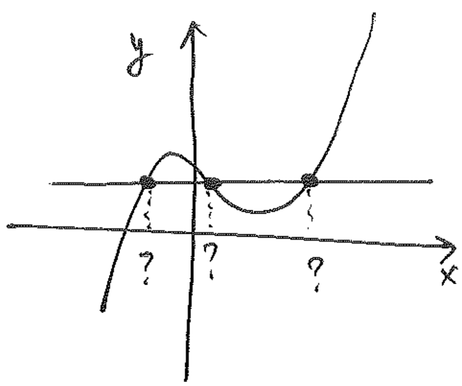


$\Rightarrow$ we see that $f^{-1}(x)$ does not pass vertical line test.

$\Rightarrow$ By the mirror argument we see that this is because the original function does not pass a "horizontal line test"

HORIZONTAL LINE TEST: To verify that the inverse of a function is unique, we check that the function satisfies the horizontal line test:

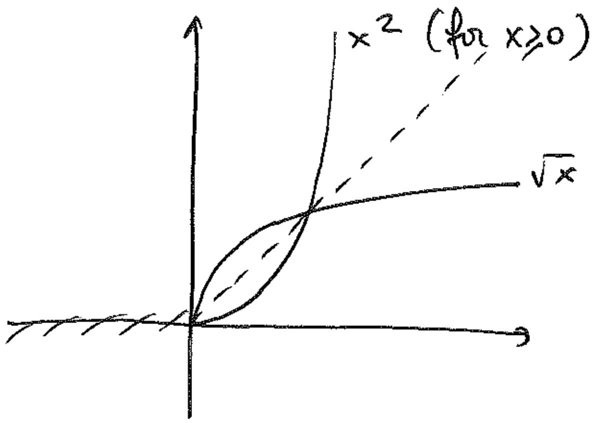


This function <u>fails</u> the horizontal line test. It has <u>no inverse</u>.

This is because there is more than one answer to the question "given y, which x does it come from".

When a function $f(x)$ does not satisfy the horizontal line test, we can often choose a smaller domain for which the inverse *is* unique.

EXAMPLE: for the function $f(x) = x^2$, we saw earlier that the inverse of $f(x) = x^2$ is defined provided we select only the interval for which $x \geq 0$. In this interval, the function $f(x)$ does satisfy the horizontal line test.



- $y = x^2$ for $x > 0$ passes the horizontal line test
- Restricting to $x > 0$ means there is indeed only one answer to "given y, which x does it come from" $\rightarrow$ it was the positive x!

4.2.4 Composition of functions

Knowing that if $y = f(x)$, then $x = f^{-1}(y)$, we can come to a rather interesting conclusion:

If $y = f(x) = f(f^{-1}(y))$ then $y = f(f^{-1}(y))$!

If $x = f^{-1}(y) = f^{-1}(f(x))$ then $x = f^{-1}(f(x))$!

While this may have seemed to be a simple game of plugging one thing into another, the notion of applying a function to another function is actually a very important mathematical concept, called *the composition of two functions.*

Given any two functions $f(x)$ and $g(x)$, we can construct the composition of the functions as either

$f(g(x))$ (apply f to g) $\Rightarrow f \circ g(x)$

or $g(f(x))$ (apply g to f) $\Rightarrow g \circ f(x)$

by plugging one expression as argument of the other

EXAMPLES:

- $f(x) = \sin(x)$, $g(x) = 4x - 1$: $f \circ g$:

$$f \circ g(x) = f(g(x)) = \sin(g(x)) = \sin(4x-1) = f(4x-1)$$

- $f(x) = \frac{1}{x^2-2}$, $g(x) = x + 1$: $f \circ g$:

$$f \circ g(x) = f(g(x)) = \frac{1}{g(x)^2-2} = \frac{1}{(x+1)^2-2} = f(x+1)$$

- $f(x) = \sqrt{1-x}$, $g(x) = x^2$: $g \circ f$:

$$g \circ f(x) = g(f(x)) = g(\sqrt{1-x}) = (\sqrt{1-x})^2 = |1-x|$$
$$= f(x)^2$$

IMPORTANT NOTE: Changing the order of the composition yields an entirely different function!

EXAMPLE: $f(x) = \sqrt{x}$, $g(x) = x^2 + 1$

- $f \circ g$: $\quad f \circ g(x) = f(g(x)) = \sqrt{x^2+1}$

- $g \circ f$: $g \circ f(x) = g(f(x)) = (\sqrt{x})^2 + 1$
  $= |x| + 1$
  $\neq \sqrt{x^2+1}$!

4.2.5 The composition of a function and its inverse

As we found out above, there are two fundamental relationships between a function and its inverse:

$$\boxed{\begin{aligned} f(f^{-1}(x)) &= x \\ f^{-1}(f(x)) &= x \end{aligned}}$$

We can check that this is true in all the examples we have seen before:

- $f(x) = 3x+2 \qquad f^{-1}(x) = \frac{x-2}{3}$

  $f(f^{-1}(x)) = 3f^{-1} + 2 = 3\left[\frac{x-2}{3}\right] + 2 = x-2+2 = x$ ✓

  $f^{-1}(f(x)) = \frac{f-2}{3} = \frac{3x+2-2}{3} = \frac{3x}{3} = x$ ✓

- $f(x) = x^2 \qquad f^{-1}(x) = \sqrt{x} \quad (\text{for } x \geq 0)$

  $f(f^{-1}(x)) = (f^{-1})^2 = (\sqrt{x})^2 = x$ ✓

  $f^{-1}(f(x)) = \sqrt{f} = \sqrt{x^2} = x$ ✓

- $s = f(r) = 6\,10^6\, r^{-0.7} \qquad r = g(s) = f^{-1}(s) = \left(\frac{s}{6\,10^6}\right)^{-\frac{1}{0.7}}$

  $\rightarrow \quad f(f^{-1}(s)) = 6\,10^6 (f^{-1})^{-0.7} = 6\,10^6\left(\left[\frac{s}{6\,10^6}\right]^{-\frac{1}{0.7}}\right)^{-0.7}$

  $= 6\,10^6 \left[\frac{s}{6\,10^6}\right]^{\frac{-0.7}{-0.7}} = \cancel{6\,10^6}\left[\frac{s}{\cancel{6\,10^6}}\right]^1 = s$ ✓

  $f^{-1}(f(r)) = \left(\frac{f(r)}{6\,10^6}\right)^{-\frac{1}{0.7}} = \left(\frac{\cancel{6\,10^6}\, r^{-0.7}}{\cancel{6\,10^6}}\right)^{-\frac{1}{0.7}}$

  $= (r^{-0.7})^{-\frac{1}{0.7}} = r^{\frac{-0.7}{-0.7}} = r^1 = r$ ✓.