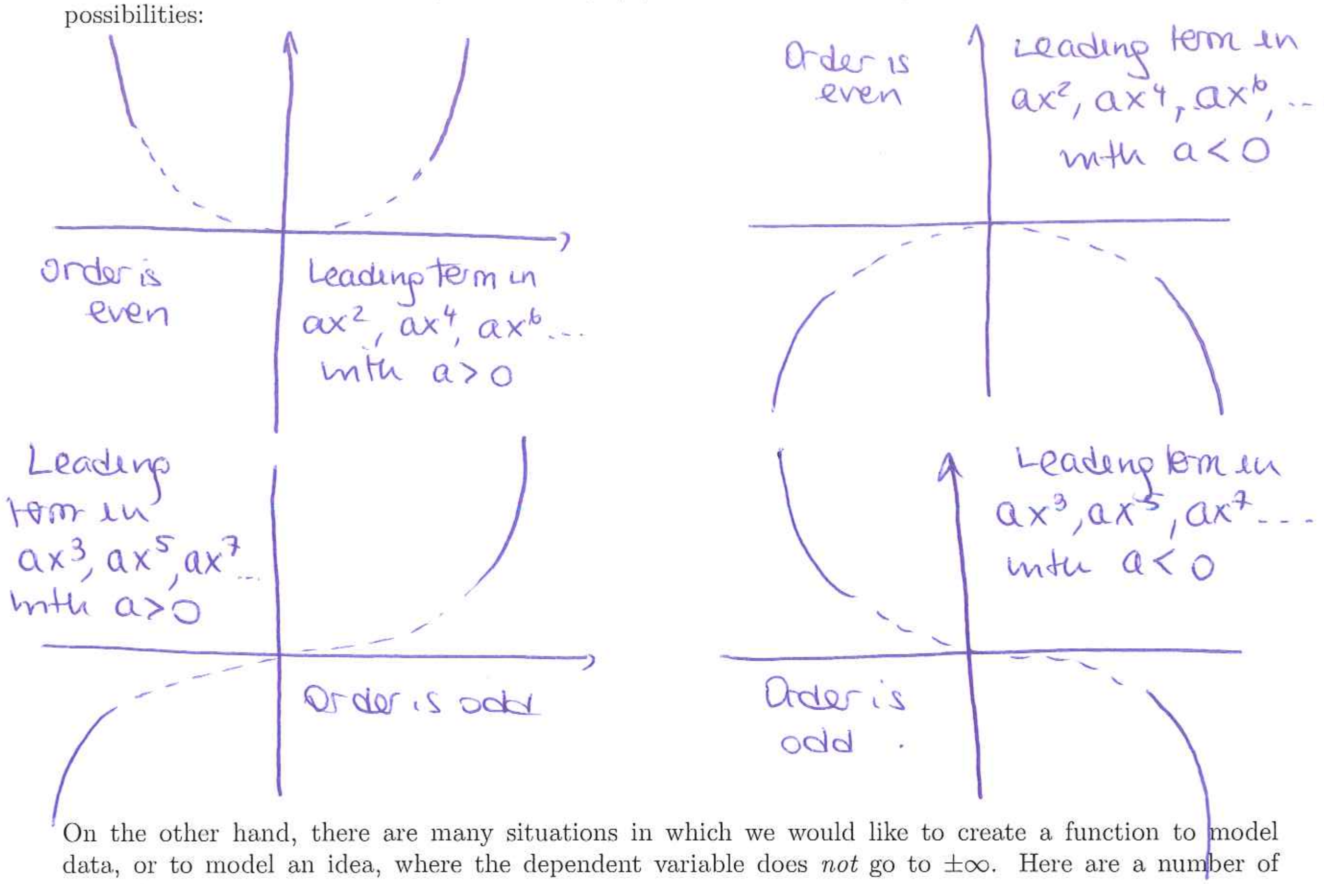
Chapter 3

Rational functions

Until now, we have learned about a general class of functions called polynomial functions, that include linear functions (i.e. polynomials of order 1) and quadratic functions (i.e. polynomials of order 2), as well as higher order polynomials. We saw that the graphs of linear functions are always straight lines, while the graphs of quadratic functions are always parabolas. The graphs of higher order polynomials are more diverse, but always either go up to $+\infty$ or down to $-\infty$ as x becomes large (either positive or negative). Indeed, since the shape of the graph for large $|x|$ looks like the leading order term, we only have four possibilities:

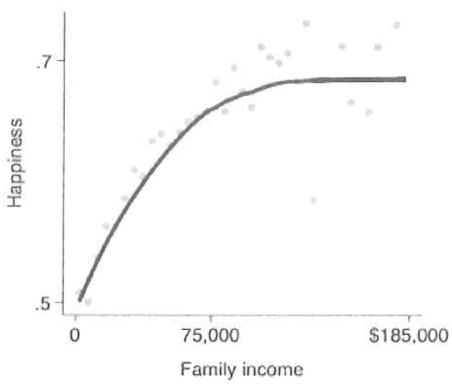


On the other hand, there are many situations in which we would like to create a function to model data, or to model an idea, where the dependent variable does *not* go to $\pm\infty$. Here are a number of examples.

3.1 Case study: *Modeling functions that are not polynomials*

3.1.1 Happiness as a function of income

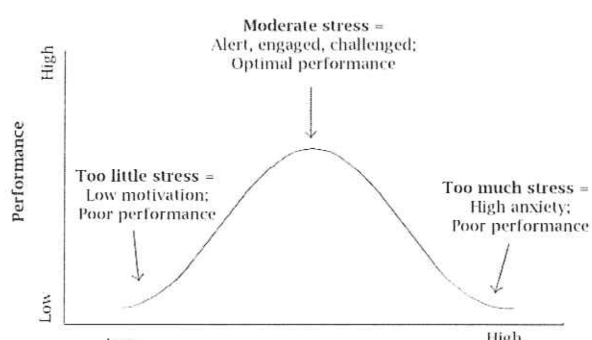
The following data is extracted from the General Social Survey website by Lane Kenworthy, in a book called 'The Good Society'. It shows a self-reported estimate of people's happiness across the US as a function of income from surveys conducted between 1972 and 2012. Happiness is quantified by the responders (about 50,000 of them!), with 0 meaning 'not too happy', 0.5 meaning 'pretty happy' and 1 meaning 'very happy'.



We need a function with the following properties
- The y-intercept is 0.5
- There is an oblique tangent at x=0
- As x → ∞ (large income) happiness tends to a constant close to 0.7
- Happiness >0 always

3.1.2 The effect of stress on performance

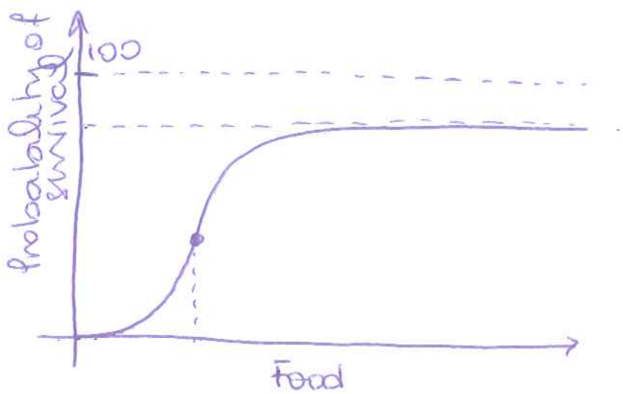
The Yerkes-Dodson law is a well-known relationship in psychology which describe how performance on a given task depends on the mental challenge associated with the task, and/or the stress of performing the task. The idea is that if there is not enough stress or mental challenge associated with the task, the performance suffers because the person is bored or not interested. Similarly, if there is too much stress or challenge, then performance also suffers. As a result, the function relating challenge to performance first increases, then reaches a maximum corresponding to the optimal level of stress/mental challenge, then decreases.



We need a function with following properties
- Performance goes to 0 as stress goes to 0
- Performance goes to 0 as stress gets very large
- There is a maximum in between
- Performance > 0 always

3.1.3 The probability of survival of offsprings as a function of available food

In ecological studies, an important factor in determining the overall survival probability of a given species in changing environmental conditions is the probability of survival to adulthood (and therefore reproductive age) of the offsprings, as a function of the available food. If too little food is available, the probability of survival to adulthood is negligible. Above a certain threshold, this probability begins to increase, and beyond a second threshold, the probability becomes essentially constant, and independent of amount of food available. This can be captured in the following graph:

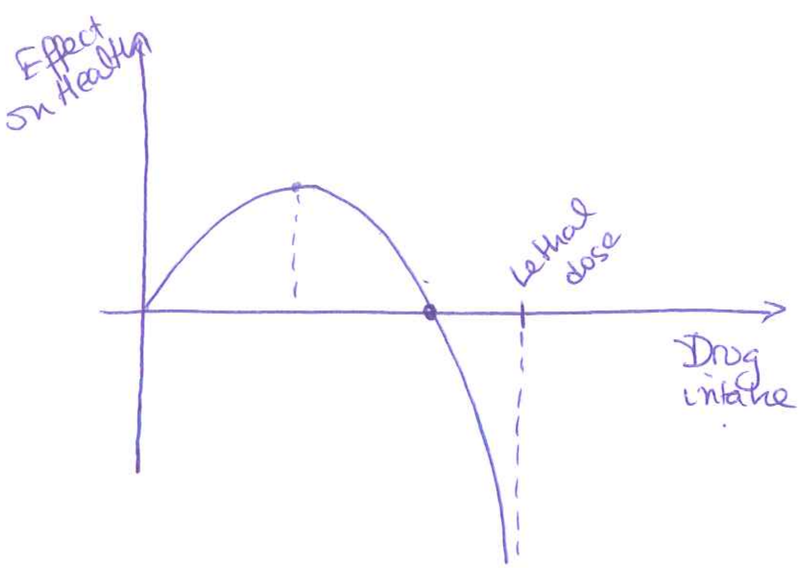


We need a function with the following properties

- probability of survival is 0 if food $\to 0$
- probability tends to a constant < 100 if food is plentifull (food $\to$ infinity)
- Probability of survival > 0 always

3.1.4 The effect of drugs

Many drugs which are designed to have a beneficial health effect can become extremely toxic, and even deadly, when taken in large quantities. This change from beneficial to toxic to lethal can be modeled with the following function:



We need a function with the following properties

- Effect on health is 0 if drug intake is 0
- Effect increases then goes back to zero at some threshold
- Effect becomes negative then goes to $-\infty$ if approach lethal dose

3.1.5 The need for more flexible functions

These examples illustrate the great diversity of functions required to model real life-examples, and this diversity simply cannot be captured by polynomials alone. For this reason, we will now learn about a greater class of functions, of which polynomials are a mere subset, and that have a much greater diversity in behavior. These functions are called Rational Functions.

3.2 Rational functions

Textbook section 3.4-3.5

3.2.1 General definition and properties of rational functions

DEFINITIONS: A rational function is the ratio of two polynomials

- In expanded form: $f(x) = \frac{a_0 + a_1 x + \cdots + a_n x^n}{b_0 + b_1 x + \cdots b_m x^m}$ ← numerator ← denominator

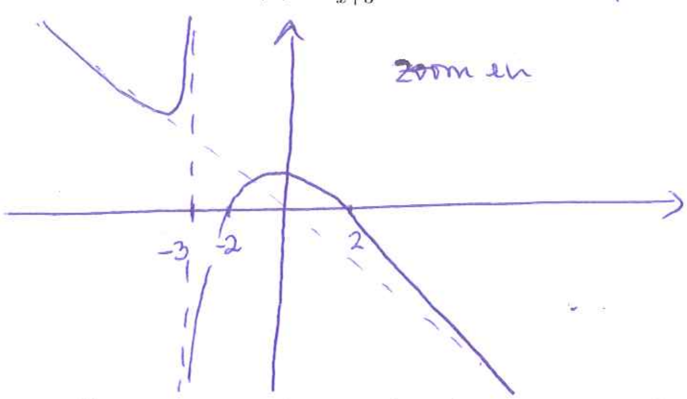
The orders of the numerator (n) & denominator (m) can be different

- In factored form, f(x) is the ratio of two fully-factored polynomials

DOMAIN OF DEFINITION: The domain of definition of a rational function contains all real numbers except the roots of the denominator

ASYMPTOTES: When the denominator of the rational function $f(x)$ goes to zero, the graph of $y = f(x)$ usually has a vertical asymptote, as in the following example:

EXAMPLE 1: $f(x) = \frac{4-x^2}{x+3}$ See Wolfram:



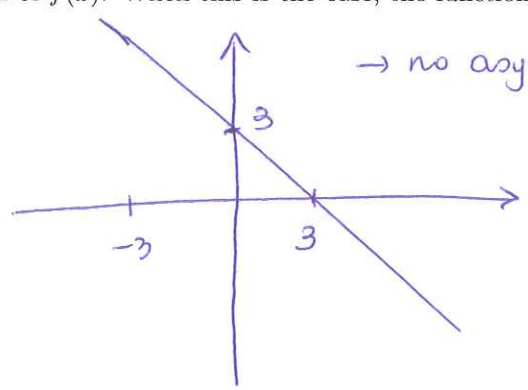
At $x=-3$, $f(x)$ is not defined. Instead we see that the graph tends to the vertical line $\boxed{x=-3}$

This is called a vertical asymptote

However, it may happen that the asymptote is "canceled out" by a root in the numerator. This situation is easy to determine from the factored form of $f(x)$. When this is the case, the function must be simplified first!

EXAMPLE 2: $f(x) = \frac{9-x^2}{x+3}$

→ no asymptote here.



$$f(x) = \frac{9-x^2}{x+3}$$

$$= \frac{(3-x)\cancel{(3+x)}}{\cancel{x+3}}$$

$$f(x) = 3-x$$

3.2.2 The behavior of rational functions for large $|x|$

In order to study the behavior of rational functions for large $|x|$ (that is, x going to $+\infty$ or x going to $-\infty$), we use the property learned in the previous chapter about the behavior of *polynomial* functions for large $|x|$:

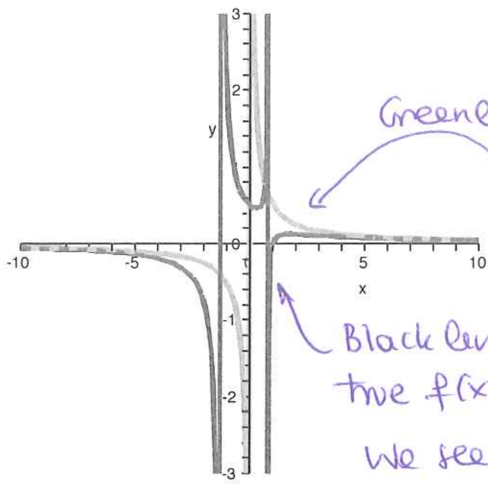
$$f(x) = a_0 + a_1 x + \dots + a_n x^n \simeq a_n x^n$$

As a result, for rational functions, $f(x) = \dfrac{a_0 + a_1 x + \cdots a_n x^n}{b_0 + b_1 x + \cdots b_m x^m} \simeq \dfrac{a_n x^n}{b_m x^m} = \dfrac{a_n}{b_m} x^{n-m}$

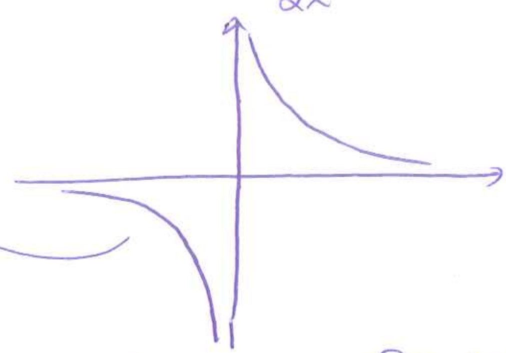
We can therefore see that rational functions can have many different kinds of behavior, depending on the order of the polynomials in the numerator relative to that of the denominator.

EXAMPLE 1: $f(x) = \frac{x-1}{2x^2+x-2}$

$$f(x) \simeq \frac{x}{2x^2} = \frac{1}{2x}$$

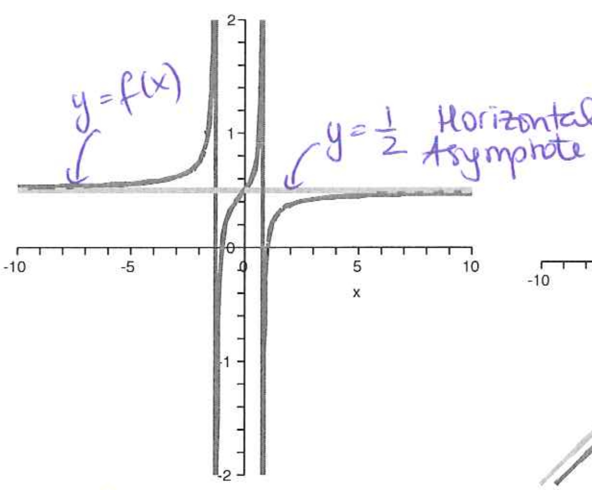


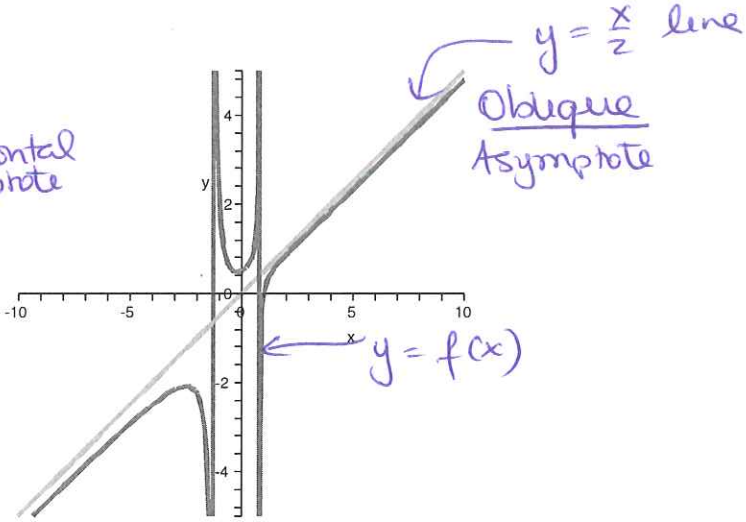
Recall that $\frac{1}{2x}$ looks like



We see $f(x) \to \frac{1}{2x}$ for large x (positive and negative)

EXAMPLE 2: $f(x) = \frac{x^2-1}{2x^2+x-2}$, $g(x) = \frac{x^3-1}{2x^2+x-2}$





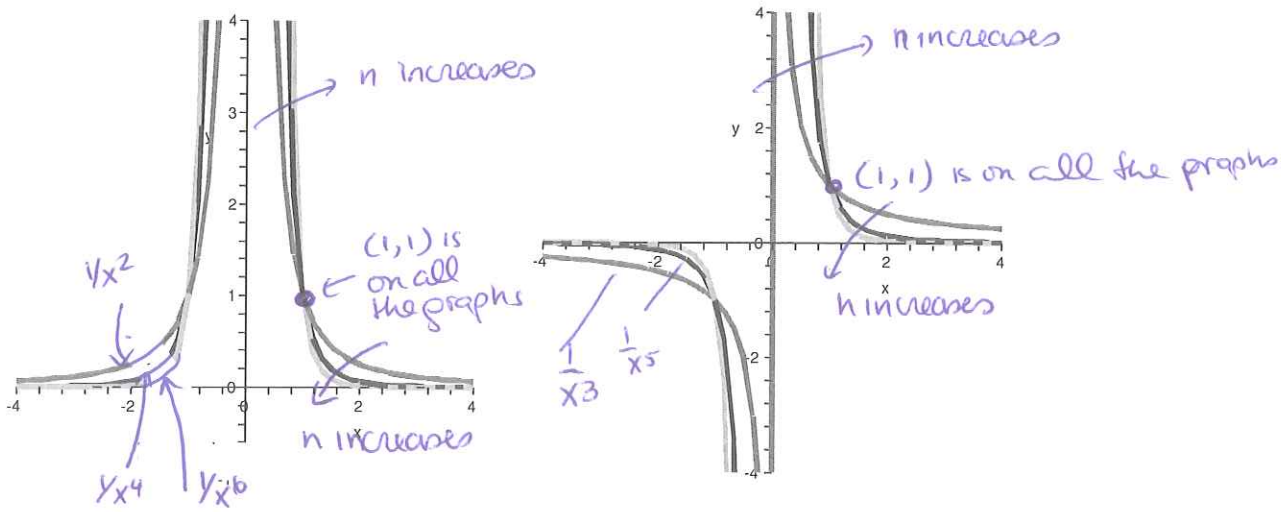
$f(x) \simeq \dfrac{x^2}{2x^2} = \dfrac{1}{2}$

We see that $f(x) \simeq \frac{1}{2}$ for large x.

$f(x) \simeq \dfrac{x^3}{2x^2} \simeq \dfrac{x}{2}$

We see $f(x) \simeq \frac{x}{2}$ for large x.

As a first step towards understanding the behavior of rational functions for large $|x|$, we therefore have to remember what the graph of functions of the kind $f(x) = x^{-n}$. As in the case of power functions, we have two different kinds of behavior depending on whether the power n is even or is odd:



As in the case of power functions with positive integer powers, the function $f(x) = x^{-n}$ is even if n is even, and is odd if n is odd:

Even powers of n $\rightarrow$ $\frac{1}{(-x)^n} = \frac{1}{x^n}$ $\rightarrow$ function even $\rightarrow$ graph symmetric about y-ax

Odd power of n $\rightarrow$ $\frac{1}{(-x)^n} = \frac{1}{-x^n}$

$\rightarrow$ function is odd

$\rightarrow$ graph is point symmetric

NOTE:

- All the graphs have a vertical asymptote at $x = 0$
- All the graphs have a <u>horizontal</u> asymptote at $y = 0$

3.2.3 Studying rational functions using signs tables

Signs tables are extremely useful tools for studying rational functions. They are used in nearly exactly the same way as for polynomial functions:

- Cast the function in a fully factored form, for both the numerator and the denominator. Simplify as needed before proceeding.
- Draw the table
- Write **all** the factors vertically on the left, including both the numerator and the denominator.

- Write **all** the points where either the numerator or the denominator goes to 0 on the top, in the correct order. Draw vertical lines below each of them.
- Determine and write the sign of each factor; write zeros where there is a root, and an infinity sign where there is an asymptote.
- Multiply the signs in each interval to determine the sign of the function.

The advantage of this method is that it can tell you very easily what the behavior of rational functions near an asymptote is, and helps graph it. We can also combine it with the information we obtained from the behavior as $|x|$ tends to infinity.

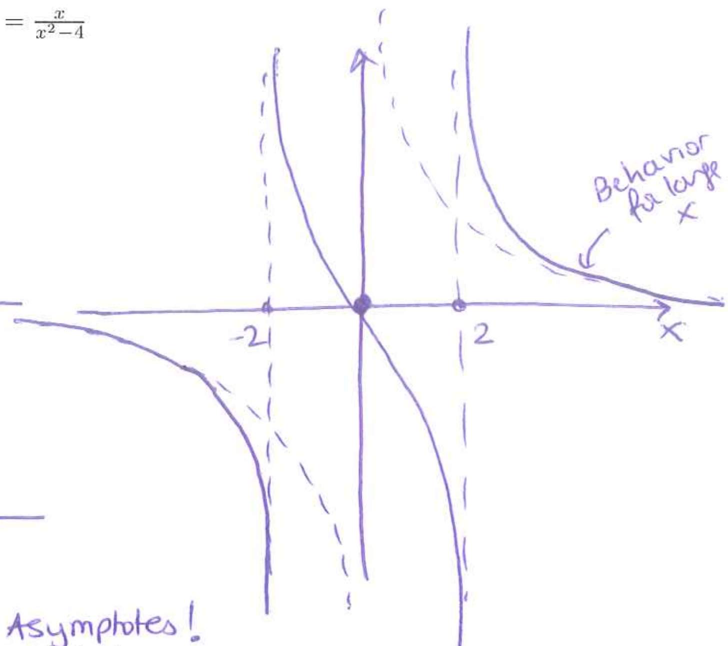
EXAMPLE 1: Study and sketch the function $f(x) = \frac{x}{x^2-4}$

For large x: $f(x) \simeq \frac{x}{x^2} \simeq \frac{1}{x}$

$$f(x) = \frac{x}{x^2-4} = \frac{x}{(x-2)(x+2)}$$

		-2		0		2	
x	−		−	0	+		+
$x-2$	−		−		−	∞	+
$x+2$	−	∞	+		+		+
	−	∞	+	0	−	∞	+

Asymptotes!



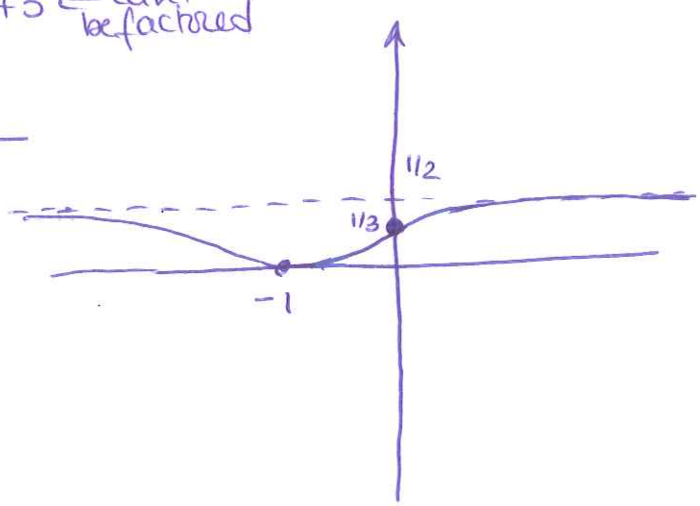
EXAMPLE 2: Study and sketch the function $f(x) = \frac{x^2+2x+1}{2x^2+3}$

$f(x) \simeq \frac{x^2}{2x^2} \simeq \frac{1}{2}$ for large x

$f(0) = \frac{1}{3}$

$$f(x) = \frac{x^2+2x+1}{2x^2+3} = \frac{(x+1)^2}{2x^2+3}$$ ← can't be factored

		-1	
$(x+1)$	−	0	+
$(x+1)$	−	0	+
$2x^2+3$	+		+
	+	0	+



EXAMPLE 3: Study and sketch the function $f(x) = \frac{x^3+2x^2-3x}{x^2-2x+1}$

$f(x) \simeq \frac{x^3}{x^2} = x$ for large x

$f(0) = 0$

$$f(x) = \frac{x^3+2x^2-3x}{x^2-2x+1} = \frac{x(x^2+2x-3)}{(x-1)^2} \leftarrow ?$$

$$D = (2)^2 - 4(1)(-3) = 4+12 = 16$$

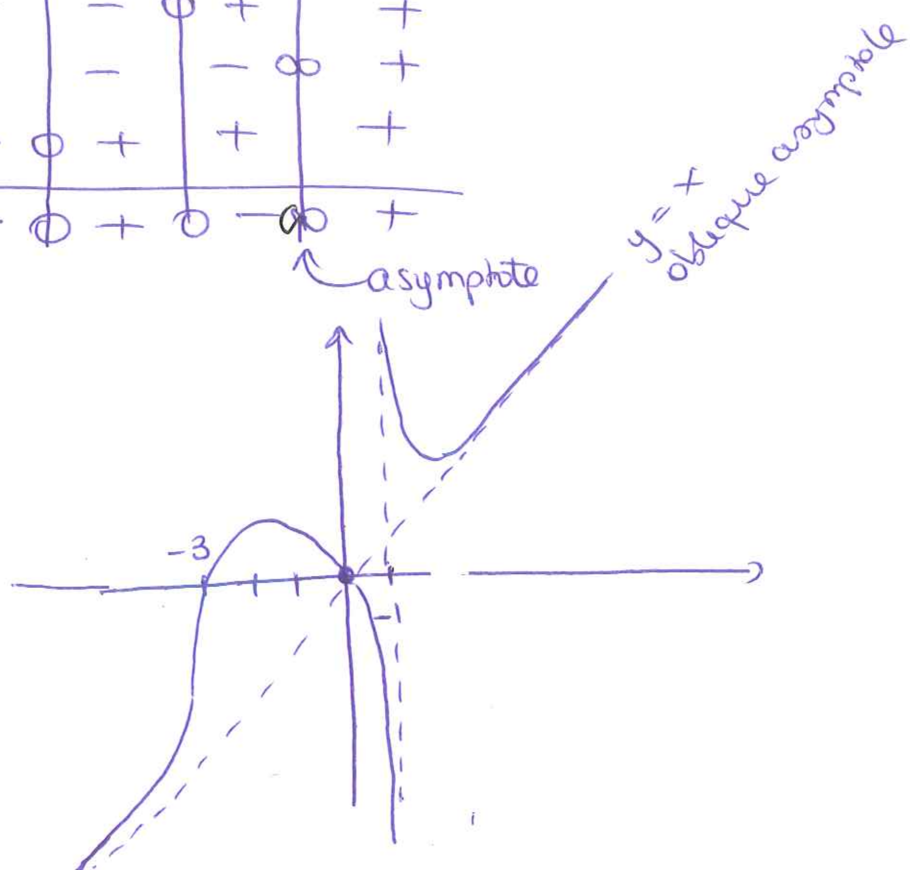
$$x_{1,2} = \frac{-2 \pm \sqrt{16}}{2(1)} = \frac{-2 \pm 4}{2} = \begin{cases} 1 \\ -3 \end{cases}$$

$$\rightarrow x^2+2x-3 = (x-1)(x+3) \text{ so}$$

$$f(x) = \frac{x\cancel{(x-1)}(x+3)}{(x-1)^{\cancel{2}}} = \frac{x(x+3)}{x-1}$$

		-3		0		1	
x	$-$		$-$	0	$+$		$+$
$x-1$	$-$		$-$		$-$	0	$+$
$x+3$	$-$	0	$+$		$+$		$+$
	$-$	0	$+$	0	$-$	∞	$+$

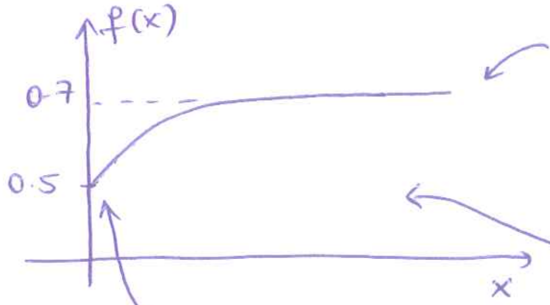
asymptote



3.3 Case study: *Modeling functions that are not polynomials*

We can now go back to our case study and try to reverse-engineer what kind of function may have a graph that looks like the data or the idea we want to model .

(a)



Note horizontal asymptote
This tells you that order of numerator & denominator have to be the same

Note no asymptotes for $x>0$
This tells you denominator has no roots

Note $f(0)=0.5$

The simplest rational function like this would be

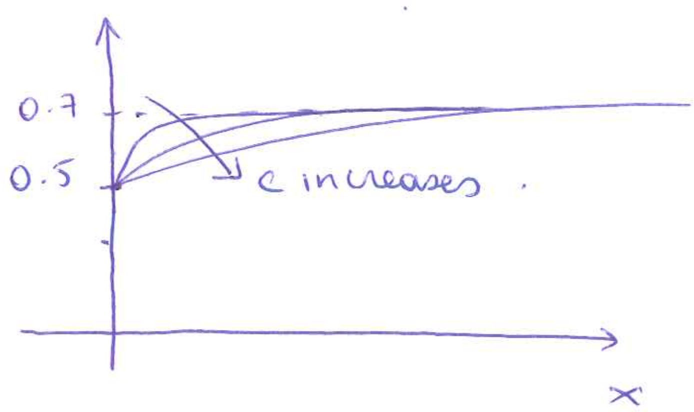
$$f(x) = \frac{ax+b}{x+c} \text{ with } c>0 \text{ so no asymptotes}$$

Also: from $f(x) \to 0.7$ for large x we guess that $f(x) = \frac{ax}{x} = \boxed{a = 0.7}$

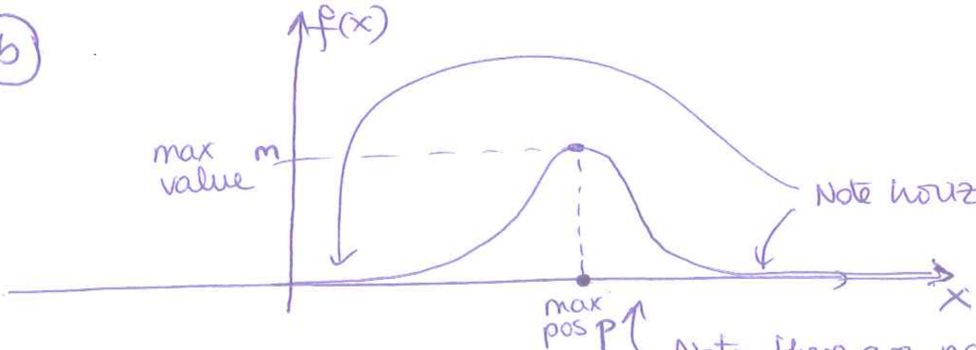
And from $f(0) = 0.5$ we get $f(0) = \frac{b}{c} = 0.5$ so $b = 0.5\,c$

$\Rightarrow$ finally $f(x) = \dfrac{0.7x + 0.5c}{x+c}$ $\to$ a 1-parameter family of curves

Let's look at them:



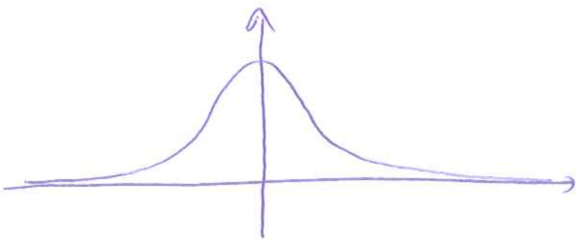
(b)



Note horizontal asymptote $y=0$

Note there are no vertical asymptotes, so denominator has no roots.

In fact note how it looks like even function but shifted to the right.



→ so let's construct an even function & shift it later.

The easiest way to construct an even function is with a polynomial / rational function that only contains <u>even</u> powers of x:

→ try $f^{even}(x) = \frac{a}{x^2+b}$ for the unshifted function

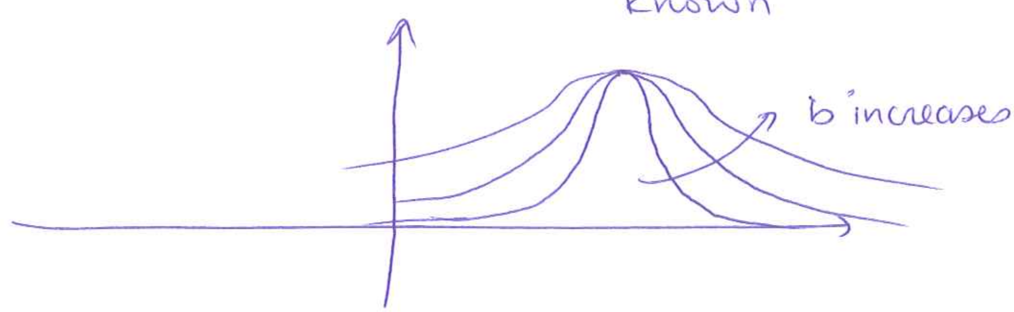
At $x=0$, $f(0) = \frac{a}{b} = m \Rightarrow a = mb$ so

$$f^{even}(x) = \frac{mb}{x^2+b}.$$

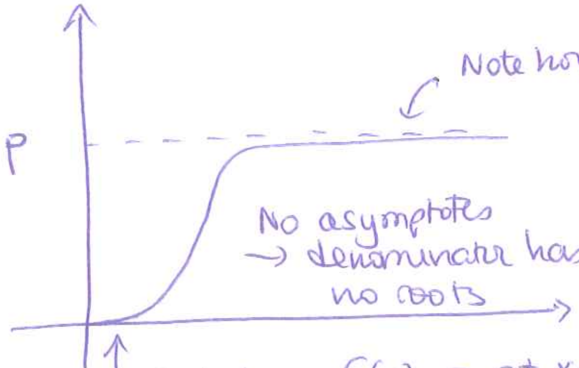
Then shift it to the right to get $f(x)$

$$f(x) = \frac{mb}{(x-p)^2+b}.$$

← a 1-parameter family as long as m & p are known



(c)



Note horizontal asymptote so need order of numerator & denominator to be the same

No asymptotes → denominator has no roots

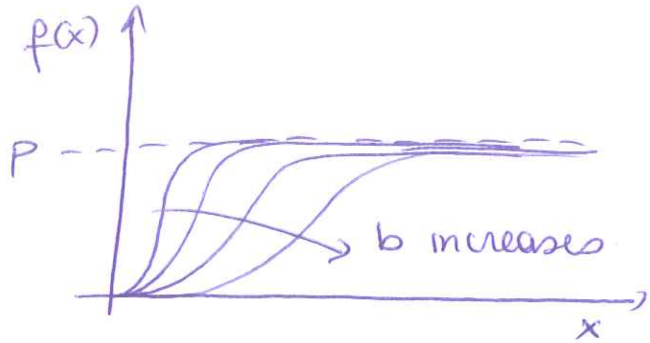
Note how $f(x)=0$ at $x=0$ and looks flat there

Note how this could be half of an even function → use only even powers.

→ try $f(x) = \dfrac{ax^2}{x^2+b}$

$f(x) \simeq \dfrac{ax^2}{x^2} \simeq a$ as $x \to \pm\infty$ & $= p$ → so $a = b$

We are left with one-parameter family of functions $f(x) = \dfrac{px^2}{x^2+b}$



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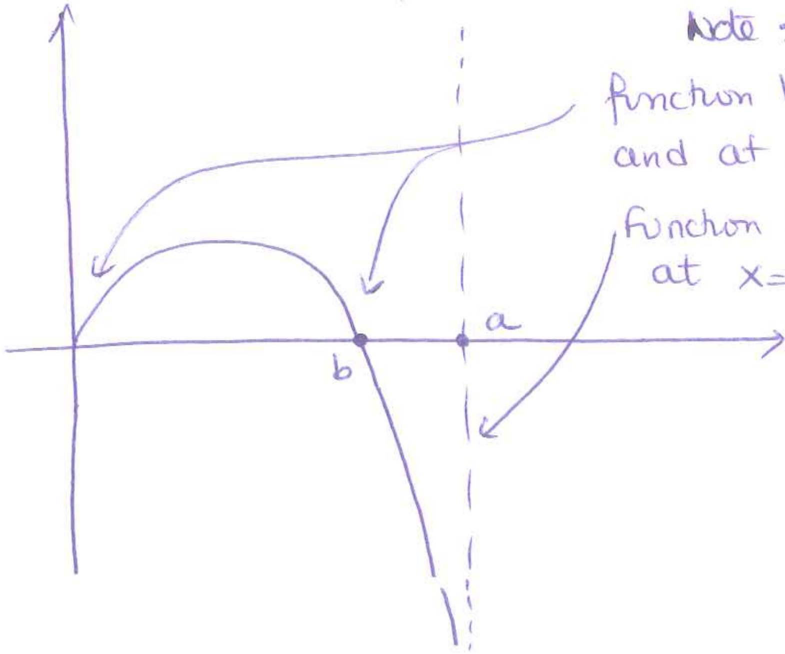

Note:
function has 2 roots at $x=0$ and at one other point b
function has an asymptote at $x=a$

→ suggests we try $f(x) = \dfrac{x(x-b)}{(x-a)}$

		0		b		a	
x	$-$	0	$+$		$+$		$+$
$x-b$	$-$		$-$	0	$+$		$+$
$x-a$	$-$		$-$		$-$	0	$+$
	$-$		$+$		$-$		$+$

domain of interest → signs match what we want

→ Given a, b known, thus fixes to function $f(x)$

To have more flexibility, we could multiply $f(x)$ by a constant c: where $c > 0$

$$f(x) = c\,\frac{x(x-b)}{x-a}$$

→ one parameter family, for given a, b