2.2 Quadratic functions

Textbook Sections 2.3, 2.4 and 2.6

2.2.1 Case study: *How to select the optimal pricing of Donelly's chocolates?*

Following on from our previous case study, we now look at how Donelly's can determine what the optimal price for their chocolates should be. The owners decide to conduct a market analysis to determine how many chocolates their customers would buy as a function of their price. After interviewing all of their customers over the course of a week, they estimate that the number of chocolates n *they would be able to sell per day as a function of their price* p *would be* $n(p) = 1000 - 200p$*. Based on this information, and using the cost analysis from the previous study, what would be optimal price for the chocolates? What is the predicted net profit at that price?*

In this case study, we ended up solving our problem graphically. However, with the right mathematical tools, we can also do it without graphs. Let's learn more about quadratics.

2.2.2 Mathematical corner: Properties of quadratic functions

A quadratic function is a special type of polynomial function. The general expression for a quadratic function is

The graph of all quadratic functions is called a *parabola*. The exact shape and position of the parabola depends on the coefficients of the quadratic. Different cases can arise:

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OPENING UP OR DOWN. Whether a parabola opens "up" or "down" can very easily be determined simply by inspection of the quadratic term ax^2 in the function.

Let's consider two examples of quadratic functions:

- $f(x) = 3x^2 - 2x - 1$
- $g(x) = -2x^2 + x + 1$

and graph them on Wolfram Alpha. We also compare their graphs with those of the functions $3x^2$ and $-2x^2$. We notice that:

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This is in fact true of all quadratics!

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Behavior near the y-axis What the parabola looks like near the $y-$ axis (i.e. when x is close to 0) can *also* very easily be determined simply by inspection of the quadratic function, but this time, of the $bx+c$ bit.

Let's consider the functions $f(x)$ and $g(x)$ again, but this time zoom in the graph near $x = 0$. We also compare their graphs with those of the functions $-2x-1$ and $x+1$. We notice that:

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Again, this is true for every quadratic!

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Vertex of a parabola and vertex form

We already saw in the previous case study an example where we were interested in finding out what the coordinates of the vertex are (in that case, where the maximum of the graph was). We can now check analytically our graphical result. Indeed,

Here are further examples. In each case, let's use the vertex coordinates formula to find the position of the vertex, then use the vertex form of the quadratic to sketch the graph of the function using basic transformations.

- $f(x) = x^2 - 1$

- $f(x) = -3x^2 + 4$

- $f(x) = x^2 - 4x + 4$

- $f(x) = x^2 + 6x + 10$

- $f(x) = -5x^2 + 30x - 46$

2.2.3 Case study: *How to select the optimal pricing of Donnelly's chocolates?*

In the previous lecture, we learned how varying the selling price of Donnelly's chocolates affects the expected number of chocolates sold in any given day, and therefore the net profit that the company can make. We then selected the pricing to maximize profit. Alternatively, the company may decide to sell the chocolates a little below this price to attract new clients. How low a price can they sell their chocolates without losing any money? Conversely, how high a price could they try to sell them without losing money?

2.2.4 Mathematical corner: Roots of quadratics

DEFINITION:

As we saw in the last lecture, some parabolas cross the $x-$axis, and some do not. Graphically, we know that there are 3 possible cases:

Since a parabola is the graph of a quadratic function, this means that some quadratic functions $f(x) = ax^2 + bx + c$ have 2 roots (i.e. 2 x-intercepts), some have one root (i.e. 1 x-intercepts), and some do not have any. In other words,

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FACTORED FORM:

The symbol $\Leftrightarrow$ implies that there is a strict equivalence relationship between the two statements "$f(x)$ can be factored" and "$f(x)$ has roots": the first implies the second, and conversely, the second implies the first. In fact, it is quite easy to check that, in both cases, the second statement implies the first. Indeed,

Whether a quadratic has one or two roots, and what the roots actually are, is therefore obvious from its factored form.

EXAMPLES

- $f(x) = 3(x-1)(x+2)$

- $f(x) = \frac{1}{2}(x-4)^2$

The interesting thing about equivalence statements in logic is that if you have one, then you also have equivalence of the opposites:

Examples

- $f(x) = x^2 + 4$

- $f(x) = x^2 + 2x + 4$

Factoring quadratics. Based on what we just saw, it would be nice to have simple tricks to

- tell us when a quadratic has roots or not,
- or equivalently, factor the quadratic if it can be factored.

As it turns out, there are a few types of quadratics that can very easily be factored:

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Examples

- $f(x) = x^2 - 2$

- $f(x) = 2x^2 - 3$

- $f(x) = x^2 + 6x + 9$

- $f(x) = 2x^2 + 4\sqrt{5}x + 10$

- $f(x) = -x^2 + 10x - 25$

On the other hand, not every quadratic is in one of these three "ideal forms". What can we do if it isn't? As it turns out, another nice trick exists in that case, and is called "The quadratic formula".

THE QUADRATIC FORMULA. Given the quadratic $ax^2 + bx + c$,

- Calculate the discriminant $D = b^2 - 4ac$
- If $D < 0$ there are no solutions to the equation $ax^2 + bx + c = 0$, and the quadratic cannot be factored.
- If $D = 0$ there is one solution to the equation $ax^2 + bx + c = 0$, $x = -\frac{b}{2a}$ and the quadratic can be factored as

- If $D > 0$ there are two solutions to the equation $ax^2 + bx + c = 0$, which are

  and the quadratic can be factored as

NOTE: The vertex of a parabola is always half-way between the roots! Indeed,

EXAMPLES:

- What are the solutions (if any) to the equation $f(x) = 2x^2 - 3x + 1 = 0$? What is the factored form of f?

- What are the solutions (if any) to the equation $f(x) = x^2 + x - 6 = 0$? What is the factored form of f?

- What are the solutions (if any) to the equation $f(x) = -2x^2 - 8x - 8 = 0$? What is the factored form of f?

- What are the solutions (if any) to the equation $f(x) = -x^2 + x - 6 = 0$? What is the factored form of f?

We can now also use this technique to solve the equation associated with our case study earlier, in a different way:

Finally, it is worth noting that this method can also help solve a few higher-order equations that can be reduced to a quadratic, as in these examples:

- What are the solutions (if any) to the equation $f(x) = x^6 - 3x^3 - 9 = 0$?

- What are the solutions (if any) to the equation $f(x) = x^4 - 2x^2 - 3 = 0$?

Graphing quadratics. Let's now recap everything we know about the graphs of quadratic functions based on their mathematical expression. Given $f(x) = ax^2 + bx + c$,

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- Given $D = b^2 - 4ac$:

Based on all this information (much of which is actually redundant) we can easily graph the parabola.

Example 1: $f(x) = 2x^2 - 3x + 1$

EXAMPLE 2: $f(x) = -x^2 - 4x - 4$

EXAMPLE 3: $f(x) = -x^2 + x - 1$

SIGN OF A QUADRATIC: As we saw at the beginning of this Section on quadratics, for large enough x (both positive and negative)

Since x^2 is always positive, this means that, for large enough x

With this in mind, we can now figure out what the sign of $f(x)$ is for all values of x, for all three possible scenarios. Indeed, the only way $f(x)$ can change sign is when the parabola actually crosses the $x-$axis, that is, when $f(x)$ has two different roots.

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This is quite useful when trying to find, for instance, the domain of definition of functions defined as the square roots of quadratics.

- $f(x) = \sqrt{x^2 - x + 6}$

- $f(x) = \sqrt{-x^2 - x + 20}$