2.3 Higher-order polynomials

Textbook sections 3.1-3.2

2.3.1 Definition and examples

DEFINITION: A polynomial function, in expanded form, is any function of the kind

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n$$

where $a_0 \dots a_n$ are real numbers and n is integer. By definition, $a_n \neq 0$

VOCABULARY:

- n is the order of the polynomial
- $a_n x^n$ is the leading term

EXAMPLES:

$f(x) = 4x^2 - 3x + 1$ order 2 leading term: $4x^2$

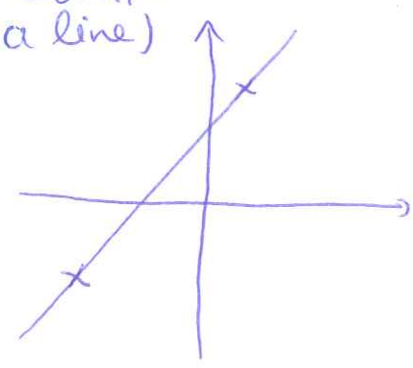
$g(y) = 1 + y + 3y^2 + y^7$: order 7, leading term y^7

$f(z) = x^2 + 3z^2 - z^4 + xz^6$: order 6, leading term xz^6 (here x is a parameter)

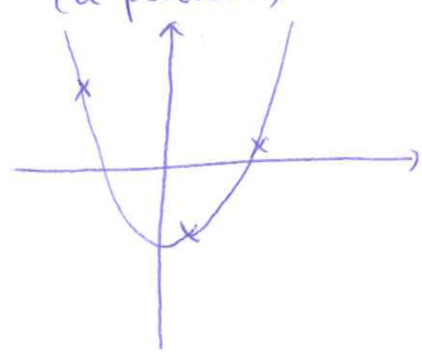
Polynomial functions are one of the most useful sets of functions in mathematics, for many reasons. They are easy to deal with (their formula is simple), they are smooth, and defined for all values of x. They are really versatile when it comes to fitting data, in the sense that it is always possible to find a polynomial function to fit, either exactly, or approximately, a finite dataset. In fact, it is not difficult to see that

To fit n data points exactly you need an order $n+1$ polynomial

2 points → 1st order
$a_0 + a_1 x$
(a line)

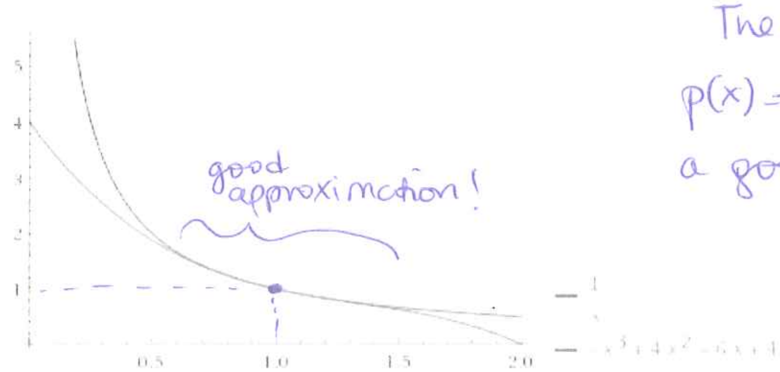


3 pts → 2nd order
$a_0 + a_1 x + a_2 x^2$
(a parabola)



4 pts → 3rd order
5 pts → 4th order
etc
⋮

In addition, you will see in Calculus that many functions can be *approximated* by polynomials near a particular value of x. For instance, consider the function $f(x) = 1/x$ (which is definitely not a polynomial), as well as the function $p(x) = 4 - 6x + 4x^2 - x^3$. Graphing them together we see that:



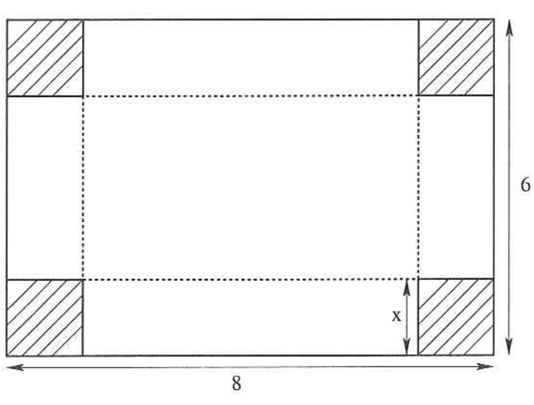
The polynomial $p(x) = 4 - 6x + 4x^2 - x^3$ is a good approximation to $f(x) = \frac{1}{x}$ near $x = 1$.

The questions of *How does one fit a polynomial function through data?* or *How does one approximate a given function by a polynomial?* are mathematically very tricky, however, and require tools for Calculus and Linear Algebra to answer.

Polynomial functions also appear naturally in many modeling problems in physics, engineering, economics, etc... In what follows, we will look first at a very simple example from geometry, then at a slightly more complicated example from Economics.

2.3.2 Case study 1: *Optimizing the volume of a cardboard box*

Suppose you have a square piece of cardboard that is 6 inches by 8 inches, and you would like to make a box out of it (without lid). A simple way to do that is to cut four squares out of the corners (as in the Figure below), fold the flaps, and tape it up. An interesting question thus arises of what size corners should we cut out to ensure that the box has the largest possible volume. Let x be the length in inches of the side of the cut-out squares.

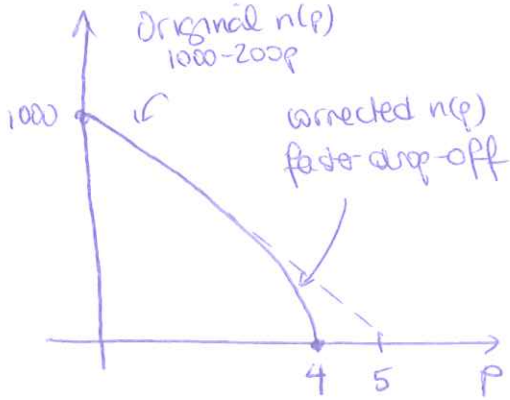


The height of the box will be x
The first side of the box will be $6-2x$ and the other side will be $8-2x$ $\rightarrow$
Volume $= x \cdot (6-2x)(8-2x)$
$(= 4x^3 - 28x^2 + 48x)$

The volume of the folded box is a polynomial (cubic) function of x! We also see that it is already factored. What we now want to find out is what this function looks like, and where its maximum (if there is one) is.

2.3.3 Case study 2: *How to select the optimal pricing of Donelly's chocolates?*

In the Case Study in the last lecture involving estimating the optimal price for Donnelly's chocolates, we used a linear relationship to describe the expected number of chocolates sold as a function of their price. However, this needs not always be the case, and in fact, that function is usually not linear. For instance, we may expect the expected number of chocolate sold to drop of faster than linearly if the price is too high, as in



To model new $n(p)$ we can use quadratic. Knowing tangent at $p=0$, we know $n(p) = ap^2 - 200p + 1000$

→ What is a? Find it by requiring $n(4) = 0$!

$$0 = a(4)^2 - 200(4) + 1000$$
$$= 16a - 800 + 1000 = 16a + 200$$
$$\Rightarrow a = -\frac{200}{16} = -12.5$$

$n(p) = -12.5p^2 - 200p + 1000$

$-500 - n(p)$

The net profit, which is given by $f(p) = pn(p)$, is now a polynomial (cubic) function. Indeed,

$$f(p) = pn(p) - 500 - n(p) = (p-1)n(p) - 500$$
$$= (p-1)(-12.5p^2 - 200p + 1000) - 500$$
$$= -12.5p^3 - 200p^2 + 1000p + 12.5p^2 + 200p - 1000 - 500$$
$$= -12.5p^3 - 187.5p^2 + 1200p - 1500$$

In this example, the function $f(p)$ is not factored. We again want to find out what this function looks like, and where its maximum (if there is one) is. In order to do that, we first have to learn a little about polynomials.

2.3.4 Mathematical corner: Basic properties of polynomials.

As we saw in the two case studies presented above, polynomials come either in *expanded form* or in *factored form*. While the expanded form is the one given in the original definition of the polynomial, we still need to formally define what a factored form is.

FORMAL DEFINITION OF FACTORED FORM: A fully factored polynomial is of the form $f(x) = (x - x_1)(x - x_2) \cdots (x - x_m) q(x)$

where
- $x_1, \ldots x_m$ are all the roots of the polynomial (some of the x_i may be the same)
- $m \leq$ the order of the polynomial (n)
- $q(x)$ is an order $n-m$ polynomial that has no roots (either $q(x) > 0$ or $q(x) < 0$)

It is not always easy to determine whether a polynomial is fully factored, or can be factored further. Sometimes, the polynomial is already obviously fully factored. Sometimes, it is partially factored, and one must decide if the remaining part can be factored further or not. Sometimes the polynomial is fully expanded, and one must start factoring it from scratch.

EXAMPLES:

- $f(x) = -(2+x)(x+3)^3$ → fully factored. roots are $x=-2$ and $x=-3$ (repeated 3 times)
- $f(x) = (x-1)(2-x^2)$ → not fully factored.
  $2-x^2 = (\sqrt{2}-x)(\sqrt{2}+x)$
  so $f(x) = (x-1)(\sqrt{2}-x)(\sqrt{2}+x)$
  $= -(x-1)(x-\sqrt{2})(x+\sqrt{2})$ → roots are $1, -\sqrt{2}, \sqrt{2}$
  $q(x) = -1 < 0$
- $f(x) = -2x(x^2-2x+1)(x+3)$ → not fully factored
  $x^2-2x+1 = (x-1)^2$
  $\Rightarrow f(x) = -2(x-0)(x-1)^2(x+3)$ → roots are 0, 1 (repeated) and -3.
  $q(x) = -2 < 0$
- $f(x) = x^3 + 2x^2 + 4x$
  $f(x) = x(x^2+2x+4)$ can x^2+2x+4 be factored?
  $D = (2)^2 - 4(1)(4) = 4 - 16 = -8$. → cannot be factored
  so $f(x) = (x-0)q(x)$ root is 0
  $q(x) = x^2+2x+4 > 0$.

In the examples presented above, it is still reasonably easy to factor the polynomial, either by finding a common factor, or by recognizing one of the standard patterns. However, there are many case in which it is not so easy. In fact, factoring high-order polynomials is notoriously difficult, and in some cases can only be done numerically. and/or finding their roots.

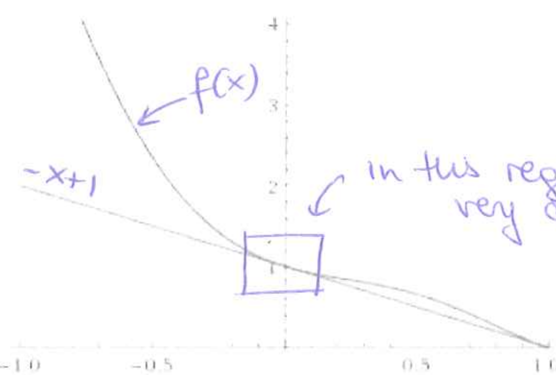
Note that a polynomial of order n has at most n roots!

Once we have both the expanded and fully factored forms of a polynomial function, we can learn a lot about its graph. For instance, from the expanded form we can deduce what the graph looks like for large and small x, just as we did for quadratics.

APPROXIMATIONS OF POLYNOMIALS FOR VERY SMALL VALUES OF x

When x is very small $f(x) = a_0 + a_1 x + \cdots a_n x^n \simeq a_0 + a_1 x$
The line $y = a_0 + a_1 x$ is tangent to the graph of $f(x)$ at $x=0$

EXAMPLE: $f(x) = x^5 - 3x^3 + 2x^2 - x + 1$

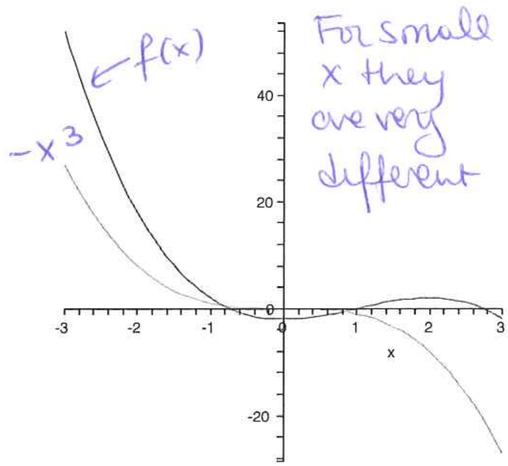


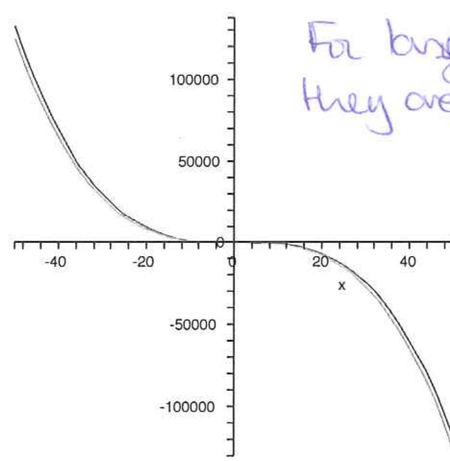
in this region the two curves are very close, but far from $x=0$ they are very different.

APPROXIMATIONS OF POLYNOMIALS FOR VERY LARGE VALUES OF $|x|$

When $|x|$ is very large (i.e x is very large positive or negative) then $f(x) = a_0 + a_1 x + \cdots a_n x^n \simeq a_n x^n$. (the leading term).

EXAMPLE: $f(x) = -x^3 + 3x^2 - 2$



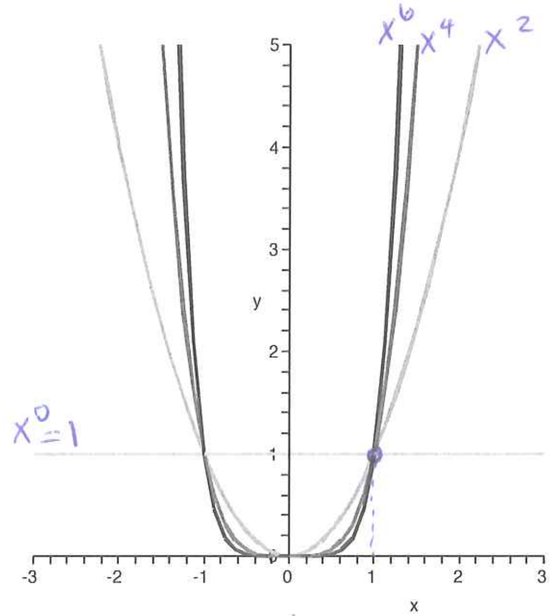


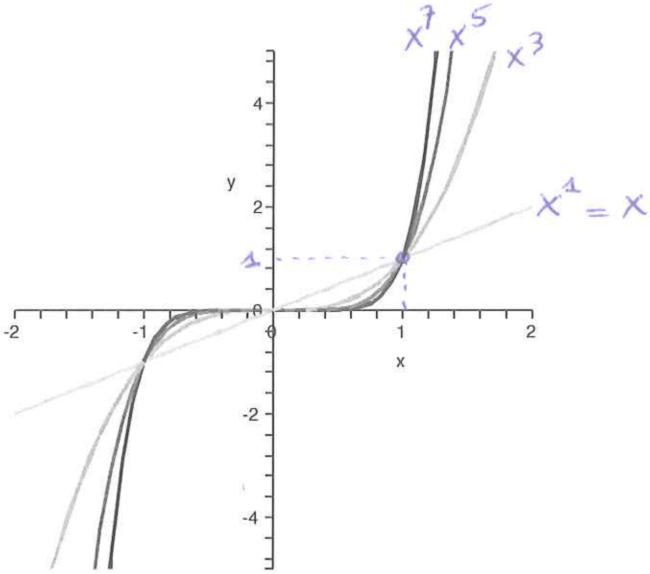
For large x, they are very similar!

In order to find out more about the overall graph of polynomial functions for large $|x|$, we therefore have to remind ourselves of the graphs of simple power functions of the kind x^n, where n is a positive integer.

POWER FUNCTIONS OF THE KIND $f(x) = ax^n$ WITH n A NATURAL NUMBER

The shape of the graphs of functions of the kind $f(x) = x^n$ depends on whether n is an even or an odd number (see above).



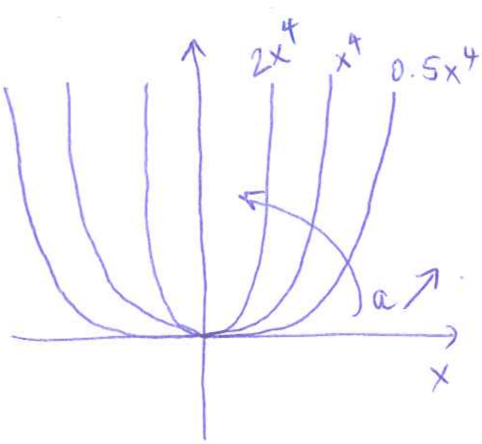


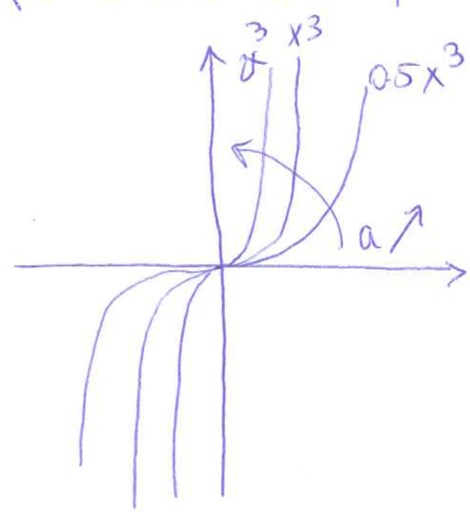
NOTE: • All functions $f(x)=x^n$ go through $(1,1)$ because $(1)^n = 1$.

- Function with even power are symmetric about y-axis, and are <u>even</u> functions: $f(-x) = (-x)^n = (-1)^n x^n = x^n$.
- Functions with odd power are point symmetric about origin and are odd functions. $f(-x) = (-1)^n x^n = -x^n$

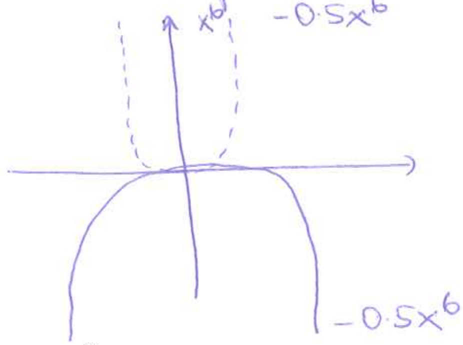
When the power is multiplied by a number a, note that $f(x) = ax^n$

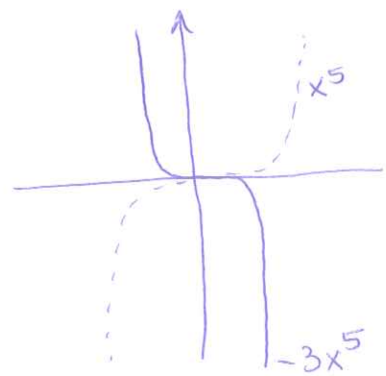
- If $a>0$ then the shape is the same, but scaled





- If $a<0$, then first we scale by $|a|$ as above, then reflect the curve about the x-axis



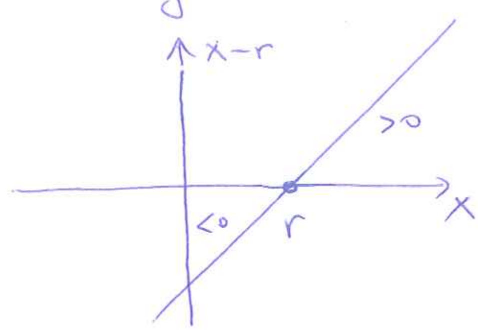


ROOTS AND SIGNS TABLES

So far we have used the expanded form to learn about the polynomial. We can also use the factored form to learn more about it, and graph the function with quite a lot of detail, but very little effort!

To do this, we first have to remember that the roots of the polynomial can be read directly from the factored form (see earlier). Then, we also have to remember that the basic factor $x - r$ (where r is one of the roots)

- is equal to zero when $x=r$ exactly
- is positive when $x>r$
- is negative when $x<r$



Finally, we also have to remember that

- The product of two positive numbers is positive
- The product of two negative numbers is positive
- The product of one positive & one negative number is negative

Using all of this, we can use a *Signs Table* to determine the sign, and therefore the overall shape, of any factored polynomial function $f(x)$.

How to draw a signs table:

- Draw the table
- Write **all** the factors vertically on the left
- Write **all** the roots horizontally on the top (in the correct order)
- Draw vertical lines below each root
- Determine and write the sign of each factor; write zeros where appropriate.
- Multiply the signs in each interval to determine the sign of the function.

assuming $x_1 < x_2 < \dots x_m$

		x_1		.....				x_m	
$x-x_1$	−	0	+		+		+		+
$x-x_2$									
⋮									
$x-x_m$	−		−		−		−	0	+
$q(x)$									

If repeated roots, then repeat the factor

← fill in signs

← put either all ⊕ or all ⊖

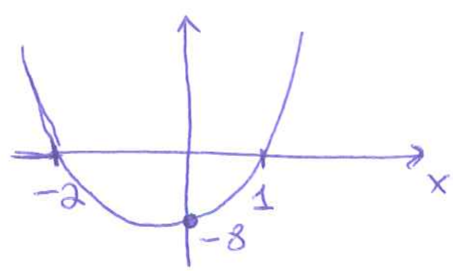
← multiply signs, voila!

Examples of use of signs tables:

Example 1: Draw a signs table and sketch the function $f(x) = 4(x-1)(x+2)$. $q(x) = 4$ here

Roots are -2 and 1

		-2		1	
$(x-1)$	−		−	0	+
$(x+2)$	−	0	+		+
4	+		+		+
	+	0	−	0	+

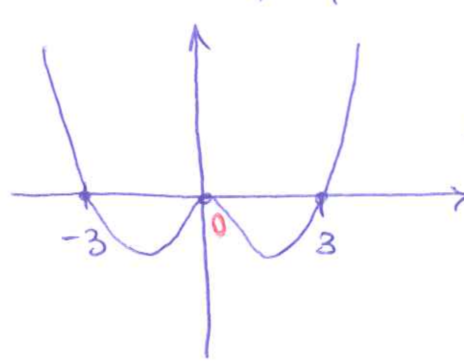


$f(0) = -8$.

Example 2: Draw a signs table and sketch the function $f(x) = x^2(x^2-9)$ $= x^2(x-3)(x+3)$

roots are 0 (twice), -3 and $+3$; $q(x) = 1$.

		-3		0		3	
x	−		−	0	+		+
x	−		−	0	+		+
$x-3$	−		−		−	0	+
$x+3$	−	0	+		+		+
	+	0	−	0	−	0	+

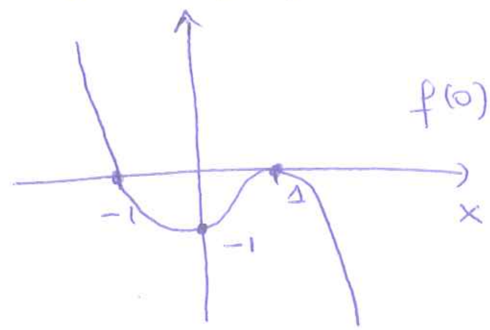


$f(0) = 0$.

EXAMPLE 3: Draw a signs table and sketch the function $f(x) = -(x+1)(x^2-2x+1)$ $= -(x+1)(x-1)^2$

roots are -1 and 1 (repeated) ; $q(x) = -1$

		-1		1	
$x+1$	$-$	0	$+$		$+$
$x-1$	$-$		$-$	0	$+$
$x-1$	$-$		$-$	0	$+$
-1	$-$		$-$		$-$
	$+$	0	$-$	0	$-$



$f(0) = -1$

EXAMPLE 4: In which interval(s) is the function $f(x) = -(2+x)(x+3)^3$ positive?

fully factored! roots are -2 and -3

		-3		-2	
$2+x$	$-$		$-$	0	$+$
$x+3$	$-$	0	$+$		$+$
$x+3$	$-$	0	$+$		$+$
$x+3$	$-$	0	$+$		$+$
-1	$-$		$-$		$-$
	$-$	0	$+$	0	$-$

In the interval $[-3,-2]$!

EXAMPLE 5: Find the domain of definition of $f(x) = \sqrt{x^5 - 2x^3 + 4x}$.

→ need radical to be > 0

$x^5 - 2x^3 + 4x = x(x^4 - 2x^2 + 4)$

Suppose $u = x^2$ then to see if $u^2 - 2u + 4$ can be factored $D = (-2)^2 - 4(4)(1) = 4 - 16 = -12 \Rightarrow$ NO –

→ $x(x^4 - 2x^2 + 4)$ is fully factored

		0	
x	$-$		$+$
x^4-2x^2+4	$+$		$+$
	$-$	0	$+$

← to find sign of $q(x)$, just evaluate $q(0)$ for instance

$\mathcal{D} = [0, +\infty)$!

Let's now go back to the two case studies, and see what we can do with the tools we have learned.

2.3.5 Case study 1: *Optimizing the volume of a cardboard box*

We have seen that the volume of the box is $V(x) = x(8-2x)(6-2x) = 4x^3 - 28x^2 + 48x$. *What can we say about this polynomial, and how can we use it to find the maximum volume of the box?*

From the expanded form we know that

- $V(x) \simeq 4x^3$ for large $|x|$
- $y = 48x$ is tangent to graph at $x = 0$

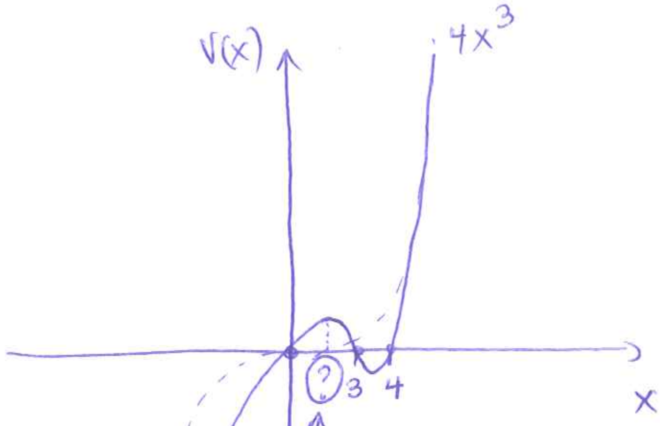
Factored form: $V(x) = x(8-2x)(6-2x)$

$$= x(-2)(x-4)(-2)(x-3)$$

$$= 4x(x-4)(x-3)$$

		0		3		4	
4	+		+		+		+
x	−	0	+		+		+
x−4	−		−		−	0	+
x−3	−		−	0	+		+
	−	0	+	0	−	0	+

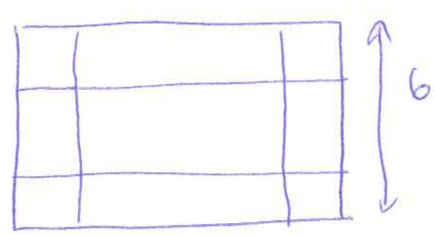
with all of this we can graph V(x)



→ we see that it has a maximum between 0 and 3 and is negative for $x < 0$ and $3 < x < 4$.

Thus maximum is what we would like.

From Wolfram α, we see that it is for $\boxed{x = 2}$



→ we can't cut out more than $x = 3$ otherwise there is nothing left ⇒ from physical argument $D = [0, 3]$

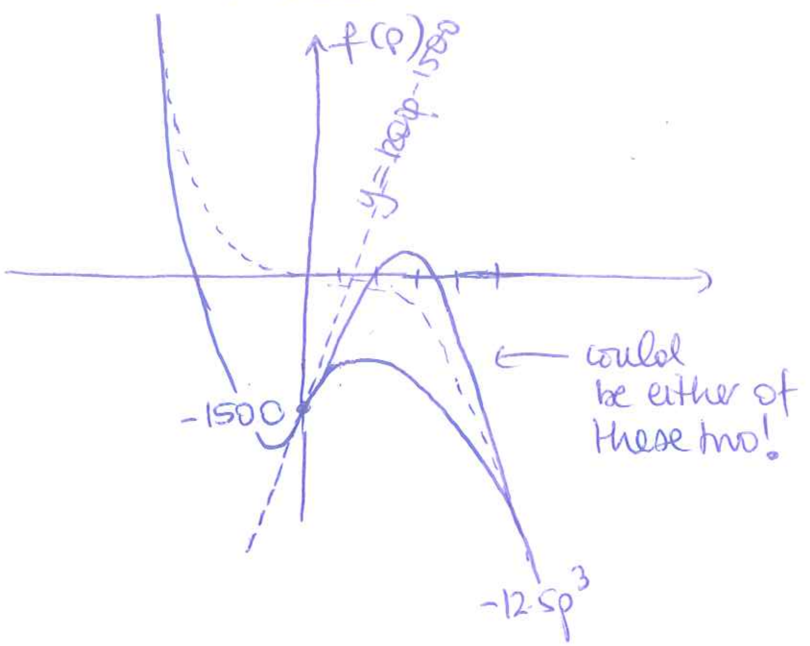
⇒ Max box volume is for $\boxed{x = 2}$

2.3.6 Case study 2: *How to select the optimal pricing of Donelly's chocolates?*

We have seen that the net profit is the following function of the price: $f(p) = (p-1)(-12.5p^2 - 200p + 1000) - 500 = -12.5p^3 - 187.5p^2 + 1200p - 1500$. *What can we say about this polynomial, and how can we use it to find the optimal price of chocolates?*

- Expanded form → • looks like $-12.5p^3$ for large $|p|$
  - tangent at $p=0$ is $y = 1200p - 1500$
- Factored form? This polynomial is not easily factored!

→ what we know so far:



→ without the factored form we don't know whether there is a price range for which the net profit is ever positive!

Graphically (using Wolfram) we see that indeed, there is. We also see that the optimal price is ~2.5

with Calculus you can actually show its $p = \sqrt{57} - 5$!

In conclusion of this Chapter, we have learned that

- Polynomial functions can be very important in mathematics, and often come up when modeling real applications.
- If we are lucky, and it so happens that a polynomial is easy to to factor, then much can be learned from Signs Tables.
- By contrast with linear and quadratic functions, however, there is no *systematic* way of studying them. For instance, factoring polynomials can be difficult even if we know roots exist, and there is no simple way of finding the local minima and/or maxima without further mathematical tools (which will be taught in Calculus).